

BULLETIN OF THE RESEARCH COUNCIL OF ISRAEL

Section C TECHNOLOGY

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Section 2
TECHNOLOGY

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REPORT ON THE QUALITY OF THE 1955 COTTON CROP

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ABSTRACT

The 1955 Israeli commercial cotton crop consisting of Acala 4-42 was quite uniform from the spinners point of view. The fibres were of good quality, similar to the American fibres of the same variety. The early picking period in the Northern part of the country yielded generally higher quality fibres with respect to the fibre strength, maturity and fibre length. The type of soil did not materially effect the fibre properties. All the results were submitted to the appropriate statistical treatments, which are briefly described in an appendix.

INTRODUCTION

The 1955 cotton crop was the first of commercial importance in Israel, and it was therefore decided to test a relatively large number of samples for the main fibre properties.

Such tests are interesting for three main reasons:

1. The assessment of possible differences in the properties of the fibres under different conditions might serve as a guide for the cotton grower in planning his future planting program.
2. The comparison of the fibres grown under local conditions to international standards.
3. The spinning mills require the highest possible quality and, perhaps even more, a uniform supply of cotton fibres. Differences in the quality of the cotton grown in various areas, on different types of soil and picked at different dates may be considerable and present serious problems in production. Accurate information on the fibre properties may facilitate the choice of cotton from the proper lots and provide data on which the spinning mills can base an efficient program of blending of the various lots obtainable in the country.

As only one variety of cotton had been planted — Acala 4-42 — in various regions of the country and on all types of soil, it is possible to assess the influence of these as well as that of the picking season on the properties of the cotton. The whole crop was grown on irrigated soil.

The cotton was ginned on three gins, Lachish serving the southern, Param the central and Beit Shean the northern region, respectively.

The different types of soil were classified as light, medium and heavy. No cotton was grown on light soils in the Lachish region. It should be emphasized that this classification is a general one and does not yet take into consideration the soil properties as determined analytically.

The first cotton was picked in August and the latest, of which samples were submitted to the Institute, was picked in January. Some cotton was picked also in February and March but no samples were submitted for testing. Most of the cotton was picked in September—October, a smaller part in November and only minor quantities from isolated fields were picked at later dates. Of the 183 samples tested, 86 came from Beit Shan, 57 from Param and 40 from Lachish. 6 were picked in August, 71 in September, 63 in October, 31 in November, 5 in December and 7 in January. 35 samples were grown on light soils, 95 on medium soils and 53 on heavy soils.

EXPERIMENTAL

Of the total crop of about 10,000 bales, a 200 gram sample was drawn from each of 183 bales and submitted for testing. The samples were drawn by the Cotton Marketing Board so as to represent proportionally the regions, types of soil and picking periods. The samples were cut out from the bales after being pressed at the gins in accordance with the standard practice demanded by the grader, were first judged by the grader and afterwards submitted to the Institute; here they were blended on a cotton blender, designed and built by the Institute.

The strength of cotton fibres was determined by means of the Pressley Strength Tester in accordance with A.S.T.M. Designation D1145—53T.¹

For determining the length of cotton fibres the array method was used. The instrument employed for the tests was the Suter-Webb in accordance with ASTM D414—54T.

The fineness of cotton fibres was tested by means of the Micronaire in accordance with ASTM D1448—54T.

The maturity of the cotton fibres was determined by means of the Causticaire method as used by the United States Department of Agriculture.

RESULTS AND DISCUSSION

In Figures 1—5 the changes of the tensile strength, maturity, fineness, U.Q.L., grade and grader's length with picking time for the three types of soil are summarized. Each point on these figures represents the calculated average of the values determined experimentally on each of the samples of the appropriate group.

In Figures 7—12 the changes in each of the properties investigated with the picking time for the 3 gins are summarized.

In the following the individual properties are discussed with reference to the statistical significance of the results obtained.

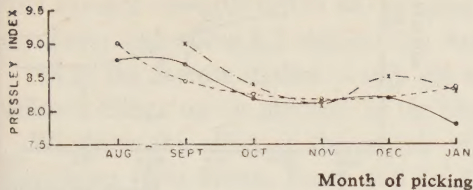


Figure 1
Changes in the tensile strength (Pressley Index, zero gauge) with the picking time of cotton fibres grown on various soils

Legend as in figure 1

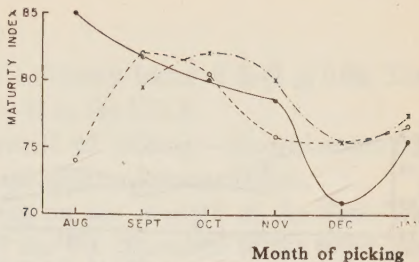


Figure 2
Changes in maturity (Caustic method) with picking time of cotton fibres grown on various soils

Legend as in figure 1

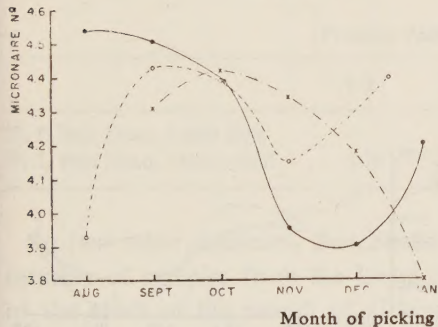


Figure 3
Changes in fineness (Micronaire number) with picking time of cotton fibres grown on various soils

Legend as in figure 1

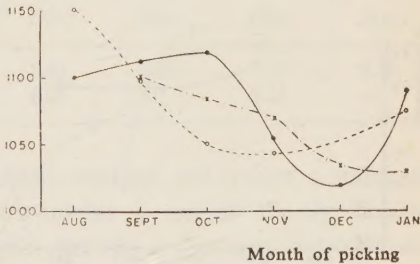


Figure 4
Changes in the upper quartile, length (U.Q.L.) with picking time of cotton fibres grown on various soils

Legend as in figure 1

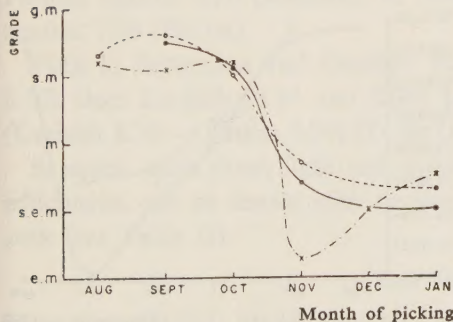


Figure 5
Changes in grade (determined by Grader) with picking time of cotton fibres grown on various soils

Legend as in figure 1

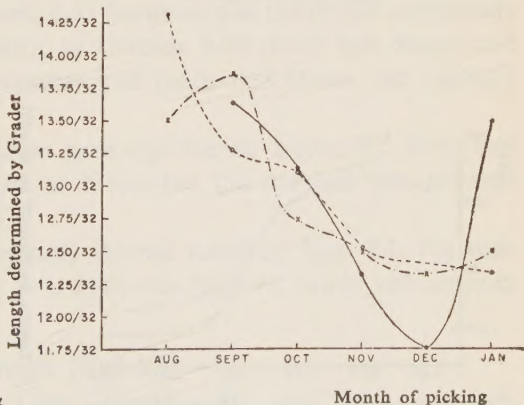


Figure 6
Changes in length (determined by Grader) with the picking time of cotton fibres grown on various soils

Legend as in figure 1

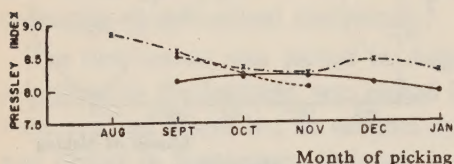


Figure 7

Changes in Pressley Index (zero gauge) with picking time of cotton fibres from 3 different gins

• — Lachish
○ - - - Param
× - . - . Beit Shean

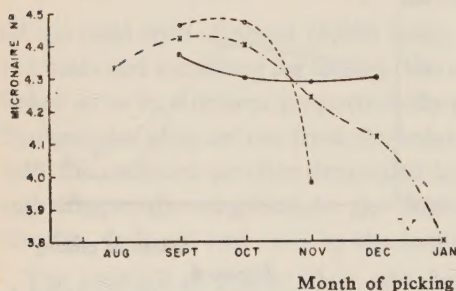


Figure 9

Changes in fineness (Micronaire number) with picking time of cotton fibres from 3 different gins
Legend as in Figure 7

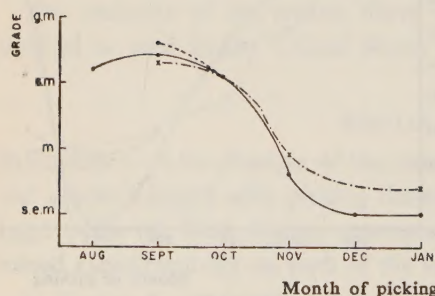


Figure 11

Changes in grade (determined by Grader) with picking time of cotton fibres from 3 different gins
Legend as in Figure 7

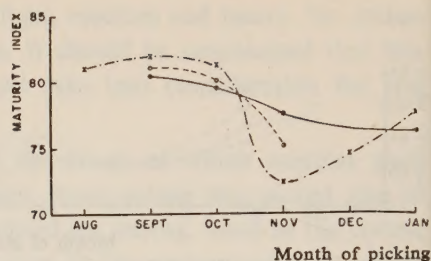


Figure 8

Changes in Maturity (Caustic method) with picking time of cotton fibres from 3 different gins
Legend as in Figure 7

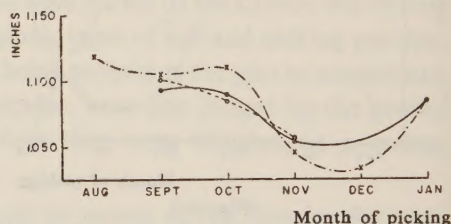


Figure 10

Changes in the upper quartile, length (U.Q.L.) with picking time of cotton fibres from 3 different gins
Legend as in Figure 7

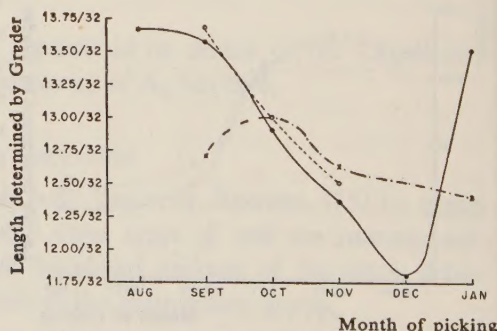


Figure 12

Changes in length (determined by Grader) with the picking time of cotton fibers from 3 different gins
Legend as in Figure 7

Strength

The average strength of the 183 samples gave a Pressley Index of 8.42 ± 0.06 . This compares with about 8.2 obtained for Acala 4—42 in the U.S.A.

All of the factors studied — soil, gin and month of picking — showed some significant effects on the strength of the fibres, expressed as Pressley Index.

Whatever the effect of the month of picking on the strength of the fibres was significant (Beit Shean, heavy and light soils) the P.I. decreased in the latter part of the season (see Table I). A similar trend was observed with the 1955 cotton crop in 5 out of 12 cotton growing states in the United States, while in none of the States a statistically significant trend of the opposite sense could be observed².

TABLE I
Pressley Index by month of picking

	<i>Aug.</i>	<i>Sept.</i>	<i>Oct.</i>	<i>Nov.</i>	<i>Dec.</i>	<i>Jan.</i>
P. I. Beit Shan, Light Soil	—	9.09	8.61	8.07	8.48	8.29
P. I. Beit Shan, Heavy Soil	8.76	8.68	8.28	8.5	8.21	—

In two other instances, Beit Shean and Lachish, medium soil, where a sufficient number of samples from the Israel crop were available to test for the significance of the effect of the month of picking, one series showed a decrease, the other an increase in the P.I., both being statistically non-significant.

It appears (see Table II) that for each of the seasons of picking for which the effect of the gin on the P.I. could be studied the P.I. of cotton from the Beit Shean area was among the higher ones, independently of the types of soil. Thus, for instance, samples from Beit Shean showed P.I.'s of 9.09 and 8.63 (medium and heavy soil respectively against 8.51 (Param) and 8.14 (Lachish); in October 8.30 (both Beit Shean and Param) against 8.20 (Lachish); in November 8.02 (both Beit Shean and Lachish) against 7.96 (Param).

Both in September and October, Param gave significantly higher P.I. (8.51 and 8.30) than Lachish (8.14 and 8.20) but in November the position was reversed (Lachish 8.20 — Param 7.96) (Table II).

Samples taken from light soil consistently showed relatively high P.I. No such conclusion can be drawn with respect to a difference between heavy and medium soils (see Table II).

Fibre Length (U.Q.L.)

The average U.Q.L. for 183 samples was 1.089 ± 0.0052 or $1\frac{3}{32}$ app. This corresponds to U.S.A. results for Acala 4—42.

The U.Q.L. decreased consistently as the season progressed. This trend was statistically highly significant in one instance (Beit Shean medium soil) and slightly significant in one other case (Beit Shean light soil). A decreasing trend in fibre length

Summary of fibre properties

Fibre property	Month of picking	Result of analysis of variance		Combinations of region and soil yielding similar fibre properties							Group Average	95% confidence limits of average for homogeneous group		
		Gins	Soils	Interaction between gins and soils		Beit Shean		Param					Lachish	
				++ +		Light	Medium	Heavy	Light	Medium				Heavy
Micro-naire Number	September	Not tested because of interaction				MH	HH	H	H	H		4.50 4.46 4.36 4.24	4.42—4.58 4.43—4.49 4.20—4.52 4.13—4.35	
	October	Bartlett's test significant				L	L	L	H	H		4.47 4.24	4.37—4.57 4.10—4.38	
	November	+	—	—		H	H	H	L	L	H	4.28 3.98	4.18—4.40 3.81—4.14	
Maturity	September	++ +	+	—		H	H	H	L	L	L	81.92 80.81	80.42—83.42 79.01—82.62	
	October	—	—	—		All combinations homogeneous							80.47	79.17—81.77
	November	+	—	—		H	H	H	L	L	L	80.28 76.76	77.05—83.51 73.73—79.79	

Notes:

- non-significant
- significant, $P < 0.05$
- ++ significant 0.05
- +++ significant $P = 0.10$
- HH — highest values of property concerned
- : H — high value of property concerned
- : L — low value of property concerned
- LL — lowest value of property concerned

(1) Analysis of variance performed for gins Beit Shean and Param only and Lachisch compared against Beit Shean and Param combined.

(2) Difference between gins and soils tested by disregarding slightly significant interaction.

from early through midseason to late crops was also observed with the 1955 crop in the U.S., but there the trend was less conspicuous than it appears to be in Israel.

No consistent effect of the gins on the U.Q.L. could be observed (see Table II): Beit Shean showed relatively high U.Q.L.'s in September and low ones in November, but in October low U.Q.L. on light and heavy soil but high U.Q.L. in medium soil. With respect to Param and Lachish differences in U.Q.L. are even less consistent, as they are always significantly affected by the type of soil too. However the effect of the soil itself is also not uniform. When heavy soils from Param yielded significantly short fibres (U.Q.L. = 1.093) but at the same time heavy soils from Beit Shean yielded short fibres (U.Q.L. = 1.042). Light and medium soils could be found to yield most inconsistently both the longer and shorter fibres, according to the region in which they had been grown.

Fineness

The average Micronaire Number for the 183 samples was 4.35 ± 0.031 comparing with 4.2 obtained in the U.S.A.

The effect of the month of picking on the Micronaire Number could be tested for only four combinations of gin and soil. In two cases the trend was upward and in two cases downward. No conclusions could therefore be drawn on the effect of the month of picking on the fineness. It might be mentioned that with the 1955 crop in the U.S. early samples tended to be coarser and late samples to be finer².

With respect to the effect of the gin and the soil on the Micronaire Number no clear cut effects could be observed either.

The only statistically significant feature was an "interaction" effect between gins and soils, i.e. that for each of the gins the effect of the soil was of a different nature.

Maturity

It should be mentioned that as the data of sowing were not identified for the samples, there is not necessarily a correlation between the date of picking and the age of the plant.

The average for all the samples gave a maturity of 80.0 ± 0.8 . This equals U.S.A. results.

The Maturity decreased consistently as the season progressed, the trend being highly significant in two instances (Beit Shean heavy soil, Lachish medium soil). Similar results were observed in the U.S.¹ where in eight out of twelve cotton growing states fibres from early samples were slightly more mature than those from midseason and late.

Beit Shean showed consistently higher maturity (September 81.9, November 80.28) than Param and Lachish (September 80.81 and November 76.76). No significant difference in maturity between samples from the latter two gins could be observed.

The effect of soil on maturity was found to be non-significant.

Grade

The grades were determined by the Marketing Board's Grader, Mr. Gefen.

To enable computations, numerical values were given to the various grades. The average for all the samples is just short of strict middling.

The difference between the picking periods was found significant: August, September and October yielded an average of better than strict middling while November, December and January gave an average of just better than strict low middling.

The difference between the soils was found not significant.

CONCLUSIONS

The 1955 crop was quite uniform from the spinners point of view. This is quite logical as only one variety was grown. The cotton was of good quality similar to that of the American Acala 4—42.

The early picking period yielded generally higher quality fibres with respect to P.I., Maturity and U.Q.L. It is important to note that the first rains fell in the middle of the picking season so that the earlier samples were picked before and the latter after the onset of the rains. Gin I gave generally better results than the other two gins, while no overall effect of difference of the soils could be observed.

Fineness, expressed as Micronaire number, averaged 4.35. Each of the three factors showed statistically significant effect on the fineness but of inconsistent direction.

The maturity index determined by the Causticaire method averaged 80.0. The more mature fibers were found in the earlier part of the season and in the Beit Shean area.

The average grade as determined by the official grader was just short of strict middling. The earlier part of the season yielding a grade somewhat better than strict middling while the latter part of the season, just better than strict low middling.

APPENDIX I

STATISTICAL ANALYSIS

S. ZACKS

The statistical analysis of the data was designed to test the significance of the respective effects of gins, soils and data of picking on various characteristics of the fibres. Results were classified according to these three factors, and graphical representations of sample means of each variable against data of picking were plotted for each combination of gins and soils.

As these diagrams showed certain trends, the significance of such trends against a null hypothesis of randomness was submitted to a nonparametric test. For this purpose a test based on Kendall's rank correlation coefficient⁴ was used and the

probability of obtaining, in the absence of any trend, a value exceeding the observed one was used as criterion.

The results of this procedure are given in Table III. Seven out of sixteen series tested by this procedure revealed statistically significant trends. As the date of picking could thus be shown to be a significant factor, the effects of gins and soils upon the measured characteristics had to be tested for each month of picking independently.

TABLE III
Effect of season on fibre properties

Fibre property	Gin	Type of soil	Direction of trend	Significance
Pressley Index	Beit Shean	Light	decreasing	+
		Medium	decreasing	—
		Heavy	decreasing	+++
	Lachish	Medium	increasing	—
Micronaire Number	Beit Shean	Light	decreasing	—
		Medium	increasing	—
		Heavy	decreasing	+++
	Lachish	Medium	increasing	—
Maturity	Beit Shean	Light	decreasing	—
		Medium	decreasing	—
		Heavy	decreasing	+++
	Lachish	Medium	decreasing	+++
Upper Quartile Length	Beit Shean	Light	decreasing	+
		Heavy	decreasing	+++
		Medium	decreasing	—
	Lachish	Medium	decreasing	—

Notes. +++ = significant at a level of $P < 0.05$
 ++ = significant at a level of $0.05 < P < 0.10$
 + = significant at a level of $P = 0.10$
 — = non significant

increasing means low values at early season high values at late season.

decreasing means high values at early season low values at late season.

The months of picking chosen for this analysis were, August, September, October and November, too few samples being available from the other months.

In order to test whether the effects of gins and of soils are significant, an analysis of variance according to a two way classification was carried out according to a model which includes an interaction factor between the two factors. This model is:

$${}^0X_{ijr} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijr}$$

where α_i stands for a gin factor at a level i ; β_j for a soil factor at a level j , γ_{ij} for an interaction between α_i and β_j and ε_{ijr} is a random error variable, normally distributed with zero mean and variance σ^2_{ij} . The assumption about the distribution of ε_{ijr} was confirmed by means of a nonparametric test concerning the Kolmogorov Smirnov d -statistic⁶ which is independent of the parameters of the normal distribution under investigation, this test confirmed our assumption.

The most serious problem which the analysis of variance had to encounter was aroused by the differences between the size of the various samples, which introduce a lack of an orthogonality of the factors α_i and β_j . Hence, a straightforward procedure of analysis of variance for a two way classification could not be applied and a more cumbersome procedure of fitting the constants α_i and β_j according to the least squares principle had to be followed.

However there remained two circumstances where even this procedure of analysis could not be followed, namely:

(a) when the sample estimates of the within class variances (σ^2_{ij}) were significantly different on the basis of Bartlett's test (Bartlett's test was in fact, significant at the 5% level only in two instances, micronaire number and U.Q.L. in September. In all other cases it was non-significant event at the 10% level).

(b) where many combinations of soils and gins were missing so that non-systematic cross classification could be obtained. In these cases, as well as whenever the analysis of variance had shown significant effects of at least one of the factors, two methods of separating sample means into homogeneous groups were applied.

These methods are due to Duncan³ and Tuckey⁷.

At last certain remarks should be made about the choice of the significance level of the tests. It is most common to take $P = 0.05$ as a significance level. However, for a given number of degrees of freedom, if P is decreased the power of the tests is also decreased. Hence, in order to keep the tests sensitive it was decided that in cases where the number of degrees of freedom is small P would be increased to 0.10.

The results of the statistical analyses are summarized in Table II.

ACKNOWLEDGMENTS

An expression of gratitude is due to the Cotton Marketing Board for supplying the samples and their interest in this work, to Mr. I. Gefen, for placing at our disposal his grading results, and to Mr. I. Ben-Zion and Miss Z. Krishmero for the devotion with which they carried out the bulk of the tests. We also wish to thank Mr. Friedlander for his active interest and for the designing of the cotton blender.

SUMMARY

Samples of Acala 4—42 cotton fibres grown under irrigation and representing the 1955 commercial crop were drawn from the three main regions, three types of soil and throughout the whole picking season and tested for their basic properties.

The fibre strength expressed as Pressley Index averaged 8.42, the highest values occurring in the early part of the season in the Beit Shean area and on light soils.

The fibre length (U.Q.L.) averaged $1\frac{3}{32}$ ". The longer fibres were obtained in the early season, but no clear effect of the region or the type of soil on the fibre length could be observed.

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REFLECTION AND TRANSMISSION OF INFRARED RAYS BY INDIAN TIMBERS

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The drying of veneers by means of infrared radiation was discussed in two earlier papers^{1,2}. It was thought of interest to determine the transmission, reflection and absorption coefficients for infrared of various Indian timbers at different moisture contents and temperatures, as only meagre data on the subject is available in the literature.

Prat³ investigated the permeability of wood to infrared rays by inserting the wood between an infrared sensitive photographic plate and the source of radiation (150W lamp at 2 m distance), an exposure period of 6 hours being given. He found *Larix*, *Abies* sp., *Picea* sp., *Alnus* and *Tilea* sp. the most permeable of the lot, i.e. they could transmit the rays through a thickness of 5—7 mm. *Pinus* sp., poplars and *Acer* sp. showed medium permeability (3—4 mm) while, *Carpinus*, *Betula*, *Fagus*, *Ulmus*, *Robinia*, *Quercus*, *Prunus*, *Juglans* and *Pterocarpus* were least transparent (0.5—2mm). Rawling⁴ and Clark⁵, using 3 mm specimens, also classified the timbers into four groups.

TABLE I

Very Transparent	Moderately transparent	Almost impermeable	Impermeable
<i>Pinus cembra</i> <i>Pinus</i> <i>Acer pseudoplatanus</i> <i>Buxus</i> <i>Fagus</i>	<i>Pinus</i> <i>Eucalyptus</i>	<i>Tectona grandis</i> <i>Swietenia</i> <i>Quercus</i>	<i>Juglans regia</i> <i>Terminalia bialata</i> <i>Diospyros</i>

Deribère⁶ studied reflection coefficients at three ranges of moisture content and his results were as follows:

TABLE II

Moisture content range	Reflection % of perfect reflection					
	Poplar		Beech		Oak	
	I. R.	W. L.	I. R.	W. L.	I. R.	W. L.
0-2	38.5	68	31.5	65	28.5	57
10-15	30.5	56	29.0	54	28	47
30-40	19	69	19	63	18.5	60

The present note records the results of tests with some Indian species.

SPECIES OF WOOD INVESTIGATED

40 species of Indian timbers varying in density from 0.2 to 1.3 g/cm³ were investigated in the air dry condition for the reflection coefficients, and several were studied in the wet state. The surfaces of the specimens were carefully prepared to the same degree of smoothness.

METHOD ADOPTED

For measuring the reflection, transmission, etc. a G.E.C. lamp (115 volt) with external parabolic reflector was used. The wood, mounted on a stand was at the focus of the lamp radiation. The reflected light was measured with the help of a Moll thermopile. For measuring the voltage of the thermopile, in the earlier experiments a milli-voltmeter was used. Later a Cambridge potentiometer with a galvanometer with an accuracy of 0.01 mm was employed. For some experiments a Zernicke galvanometer was also used. Usually the rays impinged on the surface at an angle of 30°. In a few experiments the influence of the angle of incidence was also investigated. Measurements were taken with both rock salt and glass windows for the thermopile. For penetration studies the same set up was employed. However, the rays had normal incidence and the thermopile was placed behind the specimen. 1 cm thick specimens were usually taken and after each reading they were carefully thinned down till there was just some response in the thermopile. The influence of thickness penetration was also studied in a few cases. The variation of penetration with moisture content was investigated with veneers of some species at various stages of drying. The voltage of the lamp was kept constant by means of a variac transformer and voltmeter.

RESULTS AND DISCUSSION

The percentage reflection of infrared radiation varies from 21.7% of the bright reflector in the case of ebony to 41.2% with *Cryptomeria* (see Table III). For the case of the timbers studied it appears to decrease with density (Figure 1). This is also

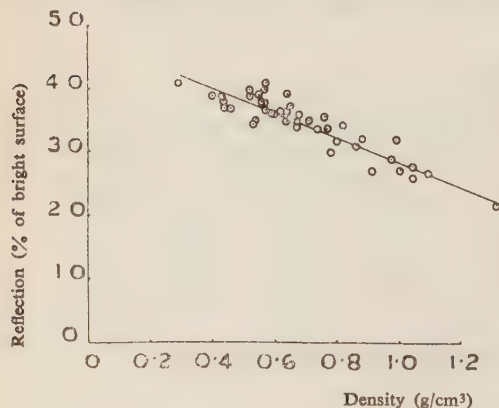


Figure 1

indicated by the results of Deribère where poplar has the higher reflection. The results of some tests in the wet condition are given in Table IV below.

It is seen that the percentage of reflection decreases with increased moisture content. This is to be expected as water is a good absorber of infrared radiation. The results are similar to those of Deribère but in the case of *Tetrameles nudiflora* the effect is minimum.

TABLE III
Relative reflection coefficients of Indian woods for infrared

	Sp. gr.	% of bright reflector		Sp. gr.	% of bright reflector
<i>Artocarpus integrifolia</i> (jack)	0.64	34.9	<i>Prunus paddam</i>	0.60	36.0
<i>Artocarpus hirsuta</i> (aini)	0.64	36.4	<i>Pentace burmanica</i> (thitoha)	0.80	31.7
<i>Abies</i> sp. (fir)	0.44	37.0	<i>Pterocarpus macrocarpus</i> (padauk)	0.88	32.2
<i>Albizia procera</i> (siris)	0.64	39.4	<i>Pinus longifolia</i> (chir)	0.44	38.1
<i>Cullenia excelsa</i>	0.59	36.2	<i>Picea morinda</i> (spruce)	0.57	39.8
<i>Cupressus torulosa</i> (cypress)	0.57	36.6	<i>Shorea robusta</i> (sal)	0.91	27.2
<i>Cedrela toona</i> (toon)	0.67	34.0	<i>Schima wallichii</i> (chilanuni)	0.74	33.8
<i>Cedrus deodara</i> (deodar)	0.54	35.0	<i>Shorea assamica</i>	0.62	36.6
<i>Calophyllum</i> sp. (poon)	0.68	35.8	<i>Swintonia floribunda</i> (taungthayet)	0.55	39.2
<i>Canarium</i> sp. (dhup)	0.57	40.7	<i>Swietenia macrophylla</i> (mahogany)	0.52	39.9
<i>Carapa moluccensis</i> (kyana)	0.99	32.0	<i>Tamarindus indica</i> (tamarind)	1.00	27.2
<i>Cryptomeria japonica</i> (suji)	0.29	41.2	<i>Terminalia paniculata</i> (kindal)	0.86	31.0
<i>Diospyros</i> sp. (ebony)	1.31	21.7	<i>Terminalia bialata</i> (chuglam)	0.57	37.6
<i>Dalbergia cultrata</i> (yindaik)	1.04	27.7	<i>Terminalia tomentosa</i> (laurel)	0.97	28.9
<i>Dalbergia latifolia</i> (rosewood)	0.78	30.1	<i>Taxus baccata</i> (yew)		32.4
<i>Dalbergia sissoo</i> (sissoo)	0.82	34.4	<i>Tectona grandis</i> (teak)	0.67	34.8
<i>Dichopsis elliptica</i> (Pali)	0.77	33.8	<i>Terminalia myriocarpa</i> (hollock)	0.52	38.8
<i>Hardwickia pinnata</i>	0.71	35.1	<i>Tetrameles nudiflora</i>	0.40	39.9
<i>Holoptelea integrifolia</i> (kanju)	0.65	37.3	<i>Xylia dolabriformis</i> (pyinkado)	1.09	26.6
<i>Juglans regia</i> (walnut)	0.53	34.6	<i>Zanthoxylum rhetsa</i>	0.43	38.8
<i>Lagerstroemia flosreginae</i>	0.76	35.6			
<i>Mangifera indica</i> (mango)	0.46	36.8			
<i>Michelia champaca</i> (champak)	0.56	38.1			
<i>Millettia pendula</i> (thinwin)	1.04	26.0			

TABLE IV

Species	% reflection	
	Dry	Wet
<i>Tetrameles nudiflora</i>	34.91	19.42
<i>Cryptomeria japonica</i>	39.20	8.17
<i>Dalbergia sissoo</i>	40.57	21.88
<i>Eleocarpus</i> sp.	—	17.92
<i>Dysoxylum malabaricum</i>	—	22.92

TABLE V

Species	Maximum thickness (mm)		Species	Maximum thickness (mm)	
<i>Cryptomeria japonica</i> (suji)	7.6	(9.2)*	<i>Homalium</i> sp.	3.4	
<i>Pinus longifolia</i> (chir)	5.6		<i>Terminalia myriocarpa</i> (hollock)	3.0	
<i>Picea morinda</i> (spruce)	5.6		<i>Gmelina arborea</i> (yemana)	3.8	
<i>Cedrus deodara</i> (deodar)	6.0	(8.5)*	<i>Zanthoxylum rhetsa</i>	3.3	
<i>Taxus baccata</i> (yew) R	4.3		<i>Dipterocarpus</i> sp. (gurjan)	4.1	
<i>Taxus baccata</i> (yew) T	5.1		<i>Ochroma</i> sp. (balsa)	7.6	
<i>Calophyllum</i> sp. (poon)	2.0		<i>Mangifera indica</i> (mango)	2.0	
<i>Terminalia bialata</i> (chuglam)	2.0	(4.5)*	<i>Tectona grandis</i> (teak)	—	(6.2)*
<i>Canarium</i> sp. (white dhup)	3.0	(4.5)*	<i>Dalbergia sissoo</i> (sissoo)	—	(4.0)*
<i>Artocarpus integrifolia</i> (jack)	3.5		<i>Juglans regia</i> (walnut)	—	(4.7)*
<i>Artocarpus hirsuta</i> (aini)	2.2				

* Readings taken with a Zernicke galvanometer.

A few experiments were carried out on the effect of the angle of incidence on reflection. Angles of incidence of 15, 20, 25 and 35° were employed. In general, the percentage reflected seems to be somewhat less at 35°. This is being further investigated. The influence of the nature of the surface is also being studied.

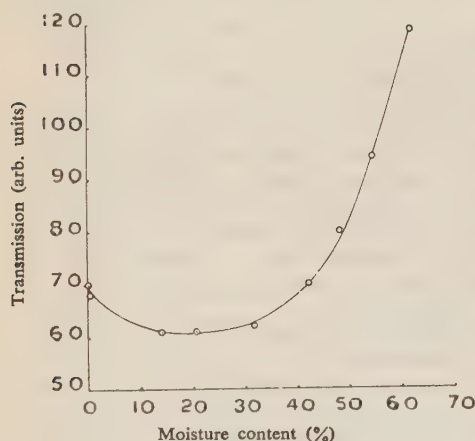


Figure 2

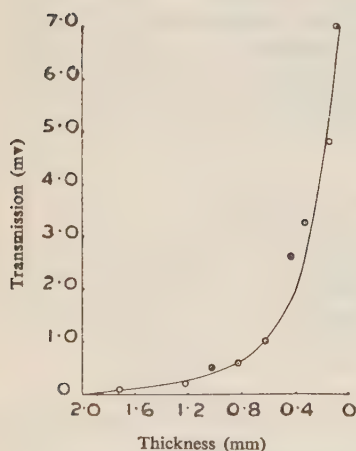
Zanthoxylum rhetsa 1/32"

Figure 3

Mango (air-dry)

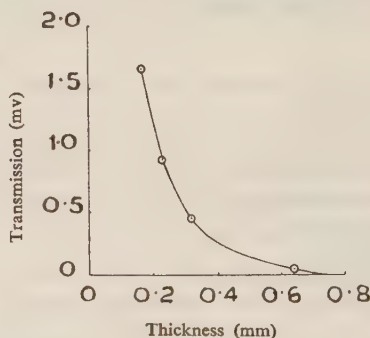


Figure 4

Balsa (air-dry)

The results of penetration studies are given in Table V above.

The above results indicate that generally conifers are more easily penetrated than hardwoods. The results of Prat³ and Rawling⁴ and Clark⁵ indicate teak, walnut and chuglam to be rather impermeable. Our results also indicate chuglam to be least permeable. Among the hardwoods balsa seems to be very permeable.

The variation of permeability with moisture content and thickness is indicated in Figures 2, 3 and 4. In a few cases a slight increase is noted again at low moisture contents.

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THE COMPLETE STRESS-STRAIN LAW IN INFINITESIMAL ELASTICITY

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ABSTRACT

It is shown that in infinitesimal strain, second order effects will be present when there is a rotation of the principal axes. The complete description then requires two new moduli in addition to the two of the classical theory.

1. The elastic state of a solid body is described by the tensor of *strain* (\mathbf{e}), to which the tensor of *stress* (\mathbf{s}) is related in a single-valued function

$$\mathbf{s} = F(\mathbf{e}) \tag{1.1}$$

or vice versa

$$\mathbf{e} = \mathfrak{F}(\mathbf{s}) \tag{1.2}.$$

In the classical theory of isotropic material, the first equation is postulated in the *linear* form

$$s_{lm} = \lambda_L \varepsilon_v \delta_{lm} + 2\mu \varepsilon_{lm} \tag{1.3},$$

where ε_{lm} is the tensor of “infinitesimal” strain, λ_L and μ are constant and ε_v is the infinitesimal cubical dilatation.

2. The concept “stress” is unequivocally defined through the primitive concepts of “force” and “area-vector”. This is not the case with regard to the concept “strain”. To define the latter we have recourse to the primitive concept “displacement gradient” (Γ). Let ${}_ix$ be the “initial”, x_l the “final” position of a particle. It is assumed that the coordinates x_l are singlevalued and continuously differentiable functions of the ${}_ix$, and *vice versa* or

$$x_l = f({}_ix) ; {}_ix = \mathfrak{f}(x_l) \tag{2.1}.$$

The *displacement* is then defined by either of the two expressions

$$u_l = x_l - \mathfrak{f}(x_l) ; {}_iu = f({}_ix) - {}_ix \tag{2.2},$$

and the *displacement gradient* by

$$\left. \begin{aligned} \Gamma_{lm} &= \partial u_l / \partial x_m = \delta_{lm} - \partial \mathfrak{f}(x_l) / \partial x_m \\ {}_{lm}\Gamma &= \partial {}_iu / \partial {}_mx = \partial f({}_ix) / \partial {}_mx - {}_{lm}\delta \end{aligned} \right\} \tag{2.3},$$

where δ_{lm} is “Kronecker’s delta”.

The strain tensor as expressed in terms of the displacement-gradient components constitutes a "measure of strain". However, this measure is not unambiguous because any isotropic symmetrical tensor function of Γ , such that it vanishes when Γ represents a rigid displacement, is a suitable measure of strain.

It should be noted that the displacement gradient is a dimensionless number. "Infinitesimal" in "infinitesimal elasticity" refers to the strain. *A strain is infinitesimal when the displacement gradient components which are the bricks making up the measure are infinitesimal.* We shall denote such a displacement gradient by γ . In this connection *infinitesimal* refers to a number which can, for the purpose of calculations carried out in a problem, be neglected when added to unity. For instance a cylinder of length l_0 , if made of steel, can be strained to a length $l = l_0 (1 + 0.001)$ so that $(l - l_0)/l_0 = 0.001$, and not more. Such strain can be considered as infinitesimal.

3. The classical theory of elasticity is "infinitesimal" in the sense mentioned before. However, its procedure "begs the question". In classical theory of elasticity the *infinitesimal strain* ε is defined by

$$\varepsilon_{lm} = \frac{1}{2} (\gamma_{lm} + \gamma_{ml}) \quad (3.1)$$

As will be shown in the present paper, the procedure of the classical theory thus suppresses second order terms which may arise even when the displacement gradient is infinitesimal, because infinitesimal second order terms can be neglected only *when the first order term is present, and in comparison with it.* For instance Love in his classical treatise defines finite strain by a measure which has as its components $e_{xx} = \partial u_x / \partial x + \frac{1}{2} [(\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2]$ etc., and proceeds to argue that "the quantities ε_{xx} ... which were called 'components of strain' in previous articles are sufficiently exact equivalents of e_{xx} , when the squares and products of such quantities as $\partial u / \partial x$ are neglected". However, consider the case when $u_x = 0$. Then the first term of Love's complete expression is reduced to zero and the term in brackets to $(\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2$. This, as has been emphasized, by one of us cannot be neglected as "infinitely small of the second order" because the first order term is absent. "Infinitely small of second order" when there is no "first order", is meaningless. The quantity $e_{xx} = \frac{1}{2} [(\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2]$, if not vanishing, cannot be neglected against zero in comparison with which it is infinitely large. There is, of course, nothing to prevent an investigator to *disregard* such a term *provided he knows of its existence.* If he does so without examining the consequences, such procedure may be dangerous. He can be likened to one driving a car on the street of a Dutch town at the side of a canal. It may be of no importance whether his forward velocity is 50 or 55 miles per hour, and he may safely neglect the additional 5 miles, but if his velocity component in the *sidewise* direction is of that "negligible" order, this will land him into the water.

The correct procedure is to solve the problem for finite strain assuming finite Γ , and to carry out the limiting process with admissible approximations assuming

infinitesimal γ in the solution. The result will reveal second order effects entirely masked by the classical procedure of linearizing at the head, effects which are amenable to experimental verification and have actually been observed. As shown in the example given before, second order effects may appear when the first order of classical infinitesimal elasticity is absent. The method for discovering such second order effects will therefore consist in solving a problem in accordance with Eqs. (3.1) and (1.3), and looking for points and directions in which some ε_{lm} vanishes. Solving the finite problem for the same point and direction, and going over to infinitesimal displacement gradients may then reveal second order effects.

4. A number of measures of finite strain have been postulated so far. When we consider the strain - ellipsoid with half-axes $\lambda(i)$, where i is a running index for i, j, k , any function of $\lambda(i)$ which vanishes for $\lambda(i) = 1$, when there is no deformation, will form a suitable measure. In terms of $\lambda(i)$ the following basic measures have been postulated by Cauchy, Green, Hencky, Almansi and Swainger in this order.*

$$\left. \begin{aligned} C_e &= \lambda - 1 \\ G_e &= \frac{1}{2}(\lambda^2 - 1) \\ H_e &= \ln \lambda = -\ln(1/\lambda) \\ A_e &= \frac{1}{2}(1 - 1/\lambda^2) \\ S_e &= 1 - 1/\lambda \end{aligned} \right\} \quad (4.1)$$

Other measures result from combinations of these. For instance a measure proposed by Wall⁴

$$W_e^1 = \frac{1}{2}(\lambda - 1/\lambda) = \frac{1}{2}(C_e + S_e) \quad (4.2)$$

and another measure proposed by Wall⁵

$$W_e^2 = (\lambda^2 - 1/\lambda) = 2G_e + S_e \quad (4.3).$$

The half axes λ in terms of the different measures are shown in the first column of Table I. Let i, j, k be the principal axes, then the ratio V_0/V , where V_0 is the initial volume of a portion of the body under consideration, and V is the volume occupied by the same particles in the strained position, is given by

$$V/V_0 = \lambda_i \lambda_j \lambda_k \quad (4.4)$$

with e_v , the cubical dilatation,

$$e_v = (V - V_0)/V_0 = V/V_0 - 1 \quad (4.5).$$

* For references compare Truesdell³.

This ratio for the different measures is shown in the second column of Table I, with I, II, III indicating the first, second and third invariant of the strain tensor, respectively.

TABLE I
Half axes of the strain-ellipsoid and volume-ratios for different strain measures

	1	2
Measure	λ	e_v
C	$1 + e$	$I + II + III$
G	$(1 + 2e)^{1/2}$	$(1 + 2I + 4II + 8III)^{1/2} - 1$
H	$\exp(e)$	$\exp(I) - 1$
A	$(1 - 2e)^{-1/2}$	$(1 - 2I + 4II - 8III)^{-1/2} - 1$
S	$(1 - e)^{-1}$	$(1 - I + II - III)^{-1} - 1$

For a general theory of elasticity we would need tensor expressions for these different measures. They have been derived by Murnaghan⁶ for $G(e)$ and $A(e)$. Hanin and Reiner⁷ have shown how to derive such expressions for any other measure, and more especially for $H(e)$. However, this is not imperative for our argumentation. What we need here are expressions for the two cases of

A) pure strain

B) simple shear,

These expressions can be derived by elementary methods following Love¹.

5. A) *Pure Strain* is one in which the principal axes do not rotate. In this case they can be chosen as axes of reference. We then have

$$\left. \begin{aligned} \lambda_i &= \partial x_i / \partial_i x = {}_i x_i \\ 1/\lambda_i &= \partial_i x / \partial x_i = {}_i x_{,i} \end{aligned} \right\} \quad (5.1)$$

and from (2.3), introducing for indices l and m the same index i , which makes $\delta_{ii} = 1$

$$\left. \begin{aligned} \lambda_i &= 1 + {}_i \Gamma \\ 1/\lambda_i &= 1 - \Gamma_i \end{aligned} \right\} \quad (5.2)$$

Eqs. (4.1) now bring forth values entered in the first line of Table II.

Now let Γ be infinitesimal, which we indicate by writing for it γ . Then from

$$1 + {}_i \gamma = 1/(1 - \gamma_i) = 1 + \gamma_i + \gamma_i^2 + \gamma_i^3 + \dots \quad (5.3)$$

there follows

$${}_i \gamma = \gamma_i \quad (5.4)^*$$

and all measures are reduced to the classical measure

$$\varepsilon_i = \gamma_i \quad (5.5).$$

* Note that this relation is valid only when no second or higher order terms are present.

TABLE II
Principal pure finite strain

	C	G
1 e_i	$i\Gamma$	$i\Gamma (1 + i\Gamma/2)$
2 e_i	$a - 1$	$\frac{1}{2} (a^2 - 1)$
3 e_j	$b - 1$	$\frac{1}{2} (b^2 - 1)$
4 e_k	$c - 1$	$\frac{1}{2} (c^2 - 1)$
5 I_o	$a + b + c - 3$	$\frac{1}{2} (a^2 + b^2 + c^2 - 3)$
6 II_o	$(a-1)(b-1) + (b-1)(c-1) +$ $+ (c-1)(a-1)$	$\frac{1}{4} [(a^2-1)(b^2-1) +$ $+ (b^2-1)(c^2-1) + (c^2-1)(a^2-1)]$
7 III_o	$(a-1)(b-1)(c-1)$	$1/8(a^2-1)(b^2-1)(c^2-1)$

	H_1	H_2
1 e_i	$\ln(1 + i\Gamma)$	$-\ln(1 - \Gamma_i)$
2 e_i	$\ln a$	$-\ln 1/a$
3 e_j	$\ln b$	$-\ln 1/b$
4 e_k	$\ln c$	$-\ln 1/c$
5 I_o	$\ln (abc)$	$-\ln (1/abc)$
6 II_o	$(\ln a)(\ln b) + (\ln b) \cdot$ $\cdot (\ln c) + (\ln c)(\ln a)$	$(\ln 1/a)(\ln 1/b) + (\ln 1/b)(\ln 1/c) +$ $+ (\ln 1/c)(\ln 1/a)$
7 III_o	$(\ln a)(\ln b)(\ln c)$	$-(\ln 1/a)(\ln 1/b)(\ln 1/c)$

	A	S
1 e_i	$\Gamma_i(1 - \Gamma_i/2)$	Γ_i
2 e_i	$\frac{1}{2} (1 - 1/a^2)$	$1 - 1/a$
3 e_j	$\frac{1}{2} (1 - 1/b^2)$	$1 - 1/b$
4 e_k	$\frac{1}{2} (1 - 1/c^2)$	$1 - 1/c$
5 I_o	$\frac{1}{2} (3 - 1/a^2 - 1/b^2 - 1/c^2)$	$3 - 1/a - 1/b - 1/c$
6 II_o	$\frac{1}{2} [(1 - 1/a^2)(1 - 1/b^2) + (1 - 1/b^2) \cdot$ $\cdot (1 - 1/c^2) + (1 - 1/c^2)(1 - 1/a^2)]$	$(1 - 1/a)(1 - 1/b) + (1 - 1/b)(1 - 1/c) +$ $+ (1 - 1/c)(1 - 1/a)$
7 III_o	$1/8(1 - 1/a^2)(1 - 1/b^2)(1 - 1/c^2)$	$(1 - 1/a)(1 - 1/b)(1 - 1/c)$

For instance, homogeneous pure strain is given by

$$\left. \begin{aligned} x_i &= a(x_i) ; x_j = b(x_j) ; x_k = c(x_k) \\ i x &= 1/a \cdot (x_i) ; j x = 1/b \cdot (x_j) ; k x = 1/c \cdot (x_k) \end{aligned} \right\} \quad (5.6)$$

where the a, b, c are constants, so that from (2.3)

$$\left. \begin{aligned} i\Gamma &= a-1 ; j\Gamma = b-1 ; k\Gamma = c-1 \\ \Gamma_i &= 1-1/a ; \Gamma_j = 1-1/b ; \Gamma_k = 1-1/c \end{aligned} \right\} \quad (5.7)$$

and from (4.2)

$$\lambda_i = a, \lambda_j = b, \lambda_k = c \quad (5.8).$$

The principal strains are shown in lines 2, 3 and 4 of Table II, followed by the invariants of the tensor. Going over to infinitesimal strain, when $\Gamma \approx \gamma$, we find for instance for ε_i , introducing from (5.2) for $\lim a = 1 + \gamma_a$, that $\frac{1}{2}(a^2 - 1) = \gamma_a(1 + \gamma_a/2) \approx \gamma_a$, when we neglect γ against unity. In this manner all measures are reduced to the same expressions, and we find the values entered in Table III.

TABLE III
Principal infinitesimal pure strains

ε_i	ε_j	ε_k	I_{ε}	II_{ε}	III_{ε}	ε_v
γ_a	γ_b	γ_c	$\gamma_a + \gamma_b + \gamma_c$	$\gamma_a \gamma_b + \gamma_b \gamma_c + \gamma_c \gamma_a$	$\gamma_a \gamma_b \gamma_c$	$\gamma_a + \gamma_b + \gamma_c$

The principal strains are accordingly of first order only, and are the same as in the classical theory.

We now consider two cases of homogeneous pure strain, namely (a) cubical dilatation, (b) linear dilatation. Their values can be found from those shown in Table III by specialisation.

6. (a) *Cubical dilatation* is given by

$$x_l = a(lx), \quad lx = 1/a \cdot x_l \quad (6.1)$$

where l stands for any index i, j, k or x, y, z . We accordingly find the values shown in Table IV.

TABLE IV
Strain-components in infinitesimal cubical dilatation

ε	I_{ε}	II_{ε}	III_{ε}	ε_v
γ	3γ	$3\gamma^2$	γ^3	3γ

(b) For *linear dilatation* in the direction k we introduce the longitudinal extension γ_l for γ_k , and the transversal contractions $-\gamma_t$ for γ_a and γ_b , and find from Table III the values entered in Table V.

TABLE V
Principal strain-components in infinitesimal linear dilatation

ε_i	ε_j	ε_k	I_{ε}	II_{ε}	III_{ε}	ε_v
$-\gamma_t$	$-\gamma_t$	γ_l	$\gamma_l - 2\gamma_t$	$-\gamma_t(2\gamma_t - \gamma_l)$	$\gamma_l \gamma_t^2$	$\gamma_l - 2\gamma_t$

7. We now go over to (B) *simple shear* as treated for instance by Love¹ (Art.37). This is kinematically given by

$$x_1 = (1x) + \Gamma(x_2), \quad x_2 = 2x, \quad x_3 = 3x \quad (7.1)$$

from which

$$\Gamma_{12} = {}_{12}\Gamma = \Gamma \tag{7.2}$$

while all other components vanish. Introducing

$$\Gamma = 2 \tan \alpha \tag{7.3}$$

Love calculates

$$\lambda_i = (1 + \sin \alpha) / \cos \alpha, \lambda_j = (1 - \sin \alpha) / \cos \alpha, \lambda_k = 1 \tag{7.4}.$$

These equations yield the expressions shown in the following Table VI.

TABLE VI
Principal Strains in Simple Shear

	C	G	H	A	S
1 e_i	$\frac{1}{2}(\sqrt{+\Gamma})-1$	$\Gamma/4 \cdot (\sqrt{+\Gamma})$	$\ln [\frac{1}{2}(\sqrt{+\Gamma})]$	$\Gamma/4 \cdot (\sqrt{-\Gamma})$	$-\frac{1}{2}(\sqrt{-\Gamma})+1$
2 e_j	$\frac{1}{2}(\sqrt{-\Gamma})-1$	$-\Gamma/4 \cdot (\sqrt{-\Gamma})$	$\ln [\frac{1}{2}(\sqrt{-\Gamma})]$	$-\Gamma/4 \cdot (\sqrt{+\Gamma})$	$-\frac{1}{2}(\sqrt{+\Gamma})+1$
3 I_e	$\sqrt{-2}$	$\Gamma^2/2$	0	$-\Gamma^2/2$	$2-\sqrt{}$
4 II_e	$2-\sqrt{}$	$-\Gamma^2/4$	$\ln [\frac{1}{2}(\sqrt{-\Gamma})] \cdot$ $\cdot \ln [\frac{1}{2}(\sqrt{+\Gamma})]$	$-\Gamma^2/4$	$2-\sqrt{}$
5	$\sqrt{=\sqrt{\Gamma^2+4}}$		$e_k=0$	$III_e=0$	$e_v=0$

However we want to know the components of the strain-tensor in respect to the system x, y, z in the final position. For this we make use of a relation proved by Love at the same place, namely that the directions of the principal axes i and j are the bisectors of the angle $\pi/2 + \alpha$ with the x -axis, and that the angle through which the principal axes i and j are turned in the course of the shear is the angle α , so that after rotation i.e. in the final position, the angle between the rotated i -axes and the x -axis is

$$\beta = \pi/4 - \alpha/2 \tag{7.5}.$$

The components of strain with respect to the system x_i or x, y, z can be derived from derived from Love's equations, Art. 12, considering that $e_{ij} = e_{jk} = e_{ki}$ vanish. Using the following equations

$$\left. \begin{aligned} e_{xx} &= e_i \cos^2 \beta + e_j \sin^2 \beta \\ e_{yy} &= e_i \sin^2 \beta + e_j \cos^2 \beta \\ e_{xy} &= (e_i - e_j)/2 \cdot \sin 2\beta \end{aligned} \right\} \tag{7.6}$$

we find the expressions entered in the following Table VII.

TABLE VII
Strain Components in Simple Shear

	C	G	H	A	S
1 e_{xx}	$-1 + \sqrt{2 + \Gamma^2}/2\sqrt{}$	$\Gamma^2/2$	$\Gamma/2\sqrt{\ln(\sqrt{+\Gamma})/(\sqrt{-\Gamma})}$	0	$1 - \sqrt{2 + \Gamma^2}/2\sqrt{}$
2 e_{yy}	$-1 + \sqrt{2 - \Gamma^2}/2\sqrt{}$	0	$-\Gamma/2\sqrt{\ln(\sqrt{+\Gamma})/(\sqrt{-\Gamma})}$	$-\Gamma^2/2$	$1 - \sqrt{2 - \Gamma^2}/2\sqrt{}$
3 e_{xy}	$\Gamma/\sqrt{}$	$\Gamma/2$	$1/\sqrt{\ln(\sqrt{+\Gamma})/(\sqrt{-\Gamma})}$	$\Gamma/2$	$\Gamma/\sqrt{}$

We now go over to infinitesimal strain. Here one must be very careful. For instance: what is $-\Gamma^2 + \sqrt{4 + \Gamma^2} - 2$, if Γ be infinitesimal? One cannot argue that with infinitesimal $\Gamma = \gamma$, γ^2 can be neglected against 4, so that the $\sqrt{}$ expression becomes 2, and the whole reduced to $-\gamma^2$. The correct procedure yields from $-\Gamma^2 + 2(1 + \Gamma^2/4)^{1/2} - 2 = -\Gamma^2 + 2(1 + \frac{1}{2}\Gamma^2/4 + \dots) - 2$ the expression $-\gamma^2 + \sqrt{4 + \gamma^2} - 2 \approx -\gamma^2 + \gamma^2/4 = -3\gamma^2/4$. We thus find the expressions entered in Table VIII.

As can be seen, all measures result in the same first order expression for the *shearing component* of the infinitesimal strain tensor, equal to the one of the classical theory. In addition, however, second order *cross-terms* appear. They are positive in the x direction, the direction of displacement (vanishing in the case of the A -measure), and negative in the y -direction, the displacement-gradient (vanishing in the case of the G -measure). Note however that a positive strain component does not indicate an extension, because in simple shear line-elements in the direction of the displacement (in our case x) do not change in length; nor does a negative strain component indicate a compression for similar reasons. Such interpretations of the classical theory are inapplicable.

TABLE VIII
Strain components in infinitesimal simple shear

	C	G	H	A	S
1 ϵ_i			$\gamma/2$		
2 ϵ_j			$-\gamma/2$		
3 I_ϵ	$\gamma^2/4$	$\gamma^2/2$	0	$-\gamma^2/2$	$-\gamma^2/4$
4 II_ϵ			$-\gamma^2/4$		
5 ϵ_{xx}	$3/8\gamma^2$	$\gamma^2/2$	$\gamma^2/4$	0	$\gamma^2/8$
6 ϵ_{yy}	$-1/8\gamma^2$	0	$-\gamma^2/4$	$-\gamma^2/2$	$3/8\gamma^2$
7 ϵ_{xy}			$\gamma/2$		
8 ϵ_v			0		

8. The very diverse expressions for strain components in different measures which we have derived in the preceding developments express one single physical phenomenon only, namely the strain ellipsoid with half axes $\lambda_i, \lambda_j, \lambda_k$. Even the rotation which is connected with simple shear is of no physical significance as long as we do not define a coordinate system fixed in space by physical means. It is therefore difficult to see what physical conditions are behind these expressions, especially when,

as we have seen, there may be present a strain component, say ε_{xx} , without the line element dx changed in length through strain. This situation can be cleared up by proceeding to the *loads* and *stresses* causing the strains and connected with them in experiments by means of which one determines the parameters appearing in equations of type (1.1) or (1.2). In these experiments specimens of certain simple shapes, mostly prisms or cylinders, are subjected to the action of certain simple loads. The mathematical problems connected with these experiments may be named *primitive problems*.

The main primitive problems refer to homogeneous deformations, caused by (i) isotropic pressure or tension of a specimen of any shape, (ii) simple pull or push of a prism, (iii) simple shear. The classical solutions of these cases are

$$(i) \quad -p = \kappa \varepsilon_v \quad (8.1)$$

which defines κ , the *bulk modulus*, and

$$(ii) \quad \sigma = E \varepsilon_l; \quad \varepsilon_l = -\nu \varepsilon_l \quad (8.2)$$

where σ is the *tensile traction*, which defines E , *Young's modulus*, and ν , *Poisson's ratio*,

$$(iii) \quad \tau = \mu \gamma \quad (8.3)$$

where τ is the *shearing traction*, which defines μ , the *shear modulus*.

When E is found from the tensile test and μ from simple shear κ and ν are calculated from

$$\left. \begin{aligned} \kappa &= E\mu/3(3\mu - E) \\ \nu &= E/2\mu - 1 \end{aligned} \right\} \quad (8.4).$$

We see that

$$E = 2\mu(1 + \nu) = 3\kappa(1 - 2\nu) \quad (8.5).$$

For mathematical convenience, but without physical significance, Lamé's constant expressed by

$$\kappa - 2/3 \mu = \lambda_1 \quad (8.6)$$

is often used, as in Eq. (1.3).

9. We now turn to the stress-strain relations.

It has been shown by Reiner⁸ that if the material is isotropic, i.e. the tensor function F of Eq. (1.1) is isotropic, it can be developed to

$$s_{lm} = F_0 \delta_{lm} + F_1 e_{lm} + F_2 e_l a e_{am} \quad (9.1)$$

where the F are scalar functions of the invariants I_e , II_e and III_e . Truesdell³ has expressed these functions as power series in the invariants, as e.g.

$$F_a = F_{a000} + F_{a100}I + F_{a200}I^2 + F_{a010}II + F_{a300}I^3 + F_{a110}I II + F_{a001}III + \dots \quad (9.2)$$

where α stands in turn for 0, 1 and 2.

Eq. (9.1) is thus replaced by

$$s_{lm} = [(F_{0000} + F_{0100}I) \delta_{lm} + F_{1000}e_{lm}] + [(F_{0200}I^2 + F_{0010}II) \delta_{lm} + F_{1100}Ie_{lm} + F_{2000}e_{l\alpha} e_{\alpha m}] + [(F_{0300}I^3 + F_{0110}II + F_{0001}III) \delta_{lm} + (F_{1200}I^2 + F_{1010}II) e_{lm} + F_{2100}I e_{l\alpha} e_{\alpha m}] + \dots \quad (9.3)$$

where the expressions within [] brackets are of the first, second and third order respectively, because the stress vanishes with the strain and the constant term F_{0000} is therefore identically equal to zero.

We now go over to infinitesimal strain assuming that the F are all of the same order.

If we compare Table VIII, we see that there are cases when I and ϵ are of second order infinitesimal, but I , II and III will be at least of first, second and third order infinitesimal, ϵ at least of first order and $\epsilon\epsilon$ at least of second order.

In the case when $F_{0100}I \delta_{lm} + F_{1000}e_{lm}$ does not vanish in the first order terms, all other expressions in [] brackets can be neglected and we have

$$s_{lm} = F_{0100}I \delta_{lm} + F_{1000} e_{lm} \quad (9.4)$$

as in classical theory. However, note that this may be the case for some definite l and m , and then (9.4) is valid only for these l, m components.

In the case when $F_{0100}I \delta_{lm} + F_{1000} e_{lm}$ vanishes in first order terms, we have

$$s_{lm} = (F_{0100}I + F_{0200}I^2 + F_{0010}II) \delta_{lm} + (F_{1000} + F_{1100}I) e_{lm} + F_{2000}e_{l\alpha} e_{\alpha m} \quad (9.5)$$

However, when I itself is of second order, $F_{0200}I^2$ can be neglected against $F_{0100}I$ and $F_{1100}I e_{lm}$ against $F_{1000} e_{lm}$. We do not know of any problem when this is not the case, nor do we know of any problem in which terms of an order higher than two appear. In this connection it may be mentioned that the invariants I and II will both vanish only when there is no strain, or $e_{lm} \equiv 0$, in which case $III \equiv 0$. We therefore omit the terms connected with the module F_{0200} and F_{1100} .

We shall now deal with primitive experiments by means of which the moduli F_{0100} , F_{0010} , F_{1000} and F_{2000} can be determined.

10. We first perform experiment (i) of §8, subjecting a specimen of the material to isotropic pressure $-p$, so that

$$s_{lm} = -p \delta_{lm} \quad (10.1).$$

Therefore from (9.4), to which (9.3) is reduced, considering Table IV,

$$-p = 3\gamma F_{0100} + \gamma F_{1000} = 3\gamma(F_{0100} + F_{1000}/3) \quad (10.2)$$

But from the same Table

$$3\gamma = \epsilon_v \quad (10.3).$$

Therefore

$$-p = (F_{0100} + F_{1000}/3) \varepsilon_v \quad (10.4).$$

Comparing (8.1), we can therefore identify

$$F_{0100} + F_{1000}/3 = K \quad (10.5).$$

11. The second experiment is (ii) the *simple pull* of §8, when

$$s_{lm} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{vmatrix} \quad (11.1)$$

Simple pull causes linear dilatation as defined in §6. We have from (9.3), which is reduced to (9.4), considering Table V,

$$\left. \begin{aligned} 0 &= F_{0100} (\gamma_l - 2\gamma_t) - F_{1000}\gamma_t \\ \sigma &= F_{0100} (\gamma_l - 2\gamma_t) + F_{1000}\gamma_t \end{aligned} \right\} \quad (11.2)$$

Introducing

$$\sigma/\gamma_l = E, \quad \gamma_t/\gamma_l = \nu \quad (11.3)$$

we find

$$\left. \begin{aligned} F_{0100} (1 - 2\nu) - F_{1000}\nu &= 0 \\ F_{0100} (1 - 2\nu) + F_{1000} &= E \end{aligned} \right\} \quad (11.4)$$

From this

$$F_{1000} (1 + \nu) = E \quad (11.5),$$

and comparing (8.5), we can therefore identify

$$F_{1000} = 2\mu \quad (11.6).$$

Going back to (10.5) we now have

$$F_{0100} = K - 2\mu/3 = \lambda_L \quad (11.7)$$

and Equ. (9.4) becomes

$$s_{lm} = \lambda_L I \delta_{lm} + 2\mu \varepsilon_{lm} \quad (11.8)$$

as in classical elasticity, Eq. (1.3)

12. We now proceed to *simple shear* as treated in §8. This is a case of plane strain when ε_{zx} , ε_{yz} and ε_{zz} vanish. The strain tensor is reduced to

$$\varepsilon_{lm} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} \quad (12.1)$$

and therefore

$$\varepsilon_{l\alpha} \quad \varepsilon_{\alpha m} = \left\| \begin{array}{cc} \varepsilon_{xx}^2 + \varepsilon_{xy}^2 & \varepsilon_{xy}(\varepsilon_{xx} + \varepsilon_{yy}) \\ \varepsilon_{xy}(\varepsilon_{xx} + \varepsilon_{yy}) & \varepsilon_{yy}^2 + \varepsilon_{xy}^2 \end{array} \right\| \quad (12.2)$$

From Table VIII this is reduced to

$$\varepsilon_{l\alpha} \quad \varepsilon_{\alpha m} = \gamma/2 \left\| \begin{array}{cc} \gamma/2 & \varepsilon_{xx} + \varepsilon_{yy} \\ \varepsilon_{xx} + \varepsilon_{yy} & \gamma/2 \end{array} \right\| \quad (12.3).$$

We now calculate s_{xy} . From (9.5), (11.6) and Table VIII we find

$$s_{xy} = \gamma/2 \cdot (2\mu + F_{1100}I) + F_{2000} \gamma/2 \cdot (\varepsilon_{xx} + \varepsilon_{yy}) + \dots \quad (12.4).$$

However, inspection of Table VIII shows us that expression $\varepsilon_{xx} + \varepsilon_{yy}$ is of the second order in every measure. The second term on the right of (12.4) can therefore be neglected against the first term. Furthermore, $F_{1100}I$ can be neglected against 2μ . The terms in the third bracket [] of (9.3) can also be neglected, and so can all following []s. We therefore have in all measures

$$s_{xy} = \mu\gamma \quad (12.5)$$

as in classical elasticity.

It is different with the normal components. We find from (9.3), taking into consideration that I, ε_{xx} and ε_{yy} are of second order, and neglecting third and higher order terms,

$$\left. \begin{aligned} s_{xx} &= \lambda_L I + 2\mu \varepsilon_{xx} - F_{0010} \gamma/4 + F_{2000} \gamma^2/4 \\ s_{yy} &= \lambda_L I + 2\mu \varepsilon_{yy} - F_{0010} \gamma^2/4 + F_{2000} \gamma^2/4 \\ s_{zz} &= \lambda_L I - F_{0010} \gamma^2/4 \end{aligned} \right\} \quad (12.6)$$

In general, in order to produce simple shear, it will therefore be necessary to exert, in addition to the classical s_{xy} of (12.5), an isotropic "cross"-pressure of the amount

$$p(c) = F_{0010} \gamma^2/4 - \lambda_L I \quad (12.7).$$

When the cross pressure $p(c)$ is absent, the volume will expand, when acted upon by shearing tractions s_{xy} . This phenomenon was described by Reynolds in the case of plastic deformations as dilatancy*. We may therefore term the factor F_{0010} modulus of dilatancy (δ).

In addition, there will be normal "cross"-stresses $s_{xx(c)}$ and $s_{yy(c)}$

$$\left. \begin{aligned} s_{yy(c)} &= 2\mu \varepsilon_{xx} + \mu_c \gamma^2 \\ s_{yy(c)} &= 2\mu \varepsilon_{yy} + \mu_c \gamma^2 \end{aligned} \right\} \quad (12.8)$$

* That such a phenomenon may be present in elasticity was suggested by Kelvin⁹.

where

$$F_{2000} = 4\mu_c \quad (12.9)$$

and μ_c is the *modulus of cross-elasticity*.

13. From the foregoing it appears that the general law of infinitesimal elasticity can be written as

$$s_{lm} = (\lambda I + \delta II) \delta_{lm} + 2\mu \varepsilon_{lm} + 4\mu_c \varepsilon_{l\alpha} \varepsilon_{\alpha m} \quad (13.1)$$

neglecting $F_{0200}I^2$ and $F_{1100}I$, and all third and higher order terms of Eq. 9.3). It is an open question whether there are cases where this approximation does not suffice. If they exist, they are not among those constituting the primitive problems. We have seen that in pure strain, second order terms are absent and classical theory reigns. However, when the strain is not pure, i.e. the principal axes are troated against a reference system formed by the external loads (x, y, z) *second order terms are present even when the strain is infinitesimal*.

For the experimental determination of μ it is usual to employ an experiment, in which the external *loads* are reduced to a torsional torque in the absence of any axial tension or pressure. As we now know, this will in general be accompanied by longitudinal and transversal positive or negative extensions.

This can be verified through Poynting's¹⁰ experiments. Poynting observed that cylinders of steel and copper in the form of long wires, when acted upon by forces equivalent to a torsional torque, are not only twisted as predicted by the classical theory of elasticity, but also elastically elongated and increased in volume. Reiner¹¹ has treated Poynting's observations in accordance with Reiner's theory of *finite* elasticity using the Almansi-measure, and realizing only at the end of his investigations that the strains in Poynting's experiments were actually infinitesimal. It is generally believed that second order effects make their appearance in finite elasticity only, but we have shown in the preceding developments that this is not so. We shall treat Poynting's observations in the light of our present investigation at another place.

CONCLUSIONS

It has been shown that when the strain of an elastic body is infinitesimal, i.e. when all components of the displacement-gradient tensor are infinitesimal, second order effects will in general be present, in accordance with the stress-strain relation.

$$s_{lm} = (\lambda I + \delta II) \delta_{lm} + 2\mu \varepsilon_{lm} + 4\mu_c \varepsilon_{l\alpha} \varepsilon_{\alpha m}$$

with four elastic moduli λ , δ , μ and μ_c . The two moduli λ and μ are those of the classical theory. In addition, however, two others are necessary for the complete description, namely δ the modulus of dilatancy and μ_c the modulus of cross-elasticity.

In infinitesimal pure strain the classical equation

$$s_m = \lambda I \delta_{lm} + 2\mu \epsilon_{lm}$$

is sufficient. While λ and μ can be determined by means of the customary experiments such as the tensile and torsion test the magnitude of δ and μ_c depends upon the measure of strain used in defining the tensor ϵ_{lm} , of which ϵ_{lm} is the infinitesimal degeneracy.

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CONSTRAINED TORSION OF AN ELLIPTICAL CYLINDER BY FORCES APPLIED ON THE LATERAL SURFACE

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ABSTRACT

St. Venant solved the free-torsion problem of a cylinder twisted by forces applied to the end sections. These classical boundary conditions occur seldom, if at all, in practical applications. A more practical case—constrained torsion—approximated by Foepl. The free torsion of cylinders by tractions applied on the lateral surface was solved by Reiner, and explicit expressions for the stress in the case of an elliptical bar twisted by constant tangential tractions were derived by him.

In the present paper the above solutions are superposed to include the case of constrained torsion of an elliptical bar fixed at one end (so that the cross-section remains plane at that end) and free at the other end from all tractions and hindrance, loaded by uniformly distributed tractions.

1. St. Venant¹ (1855) was the first to solve the problem of “free” torsion of cylinders other than circular ones, where “free” refers to the unrestricted warping of all cross-sections.

The displacements u_i take the form

$$u_x = -azy \quad (1.1)$$

$$u_y = axz \quad (1.2)$$

$$u_z = \alpha\varphi(x,y) \quad (1.3)$$

where the torsion function $\varphi(x,y)$ is harmonic throughout the cross-section R of the twisted cylinder, while its normal derivative assumes a given value on the boundary; α is the twist per unit length and z is the axis of the cylinder.

St. Venant's solution requires that:

- (a) the surface of the cylinder be entirely free from loads;
- (b) the tangential tractions s_{zx} , s_{zy} be distributed in a specific manner, which is the same in any cross-section;
- (c) the only non-vanishing components of the loads be reduced to the torque M_z .

In practical applications the assumption that the torsion is free, seldom, if ever, corresponds to the actual physical situation when, at least, one end of the cylinder is fixed and the torsion is thus “constrained”.

An extension due to A. Foepl² (1920) imposes on St. Venant's solution the requirement that one of the sections of the twisted cylinder remain plane, i.e. $u_z|_{z=0} = 0$.

Following the principle of St. Venant — “le mode d'application et de répartition des forces vers les extrémités des prismes est indifférent” — Foepl assumed that there is a local effect due to the constraint in the section $z = 0$, so that the axial displacement is

$$u_z = a\varphi(x,y)(1 - e^{-\beta z}) \quad (1.4)$$

where β is a constant. Using energy methods, he approximated this problem for a beam of elliptical cross-section (Figure 1). His expressions comply with the equations of equilibrium. They do not, however, all comply with the equations of compatibility.

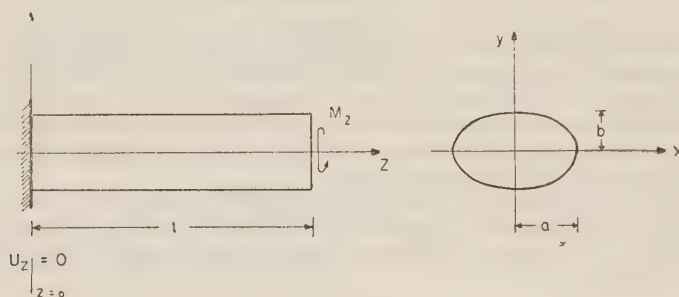


Figure 1

In all the foregoing considerations the lateral surfaces of the cylinders are free from external load.

M. Reiner^{3,4,5} (1915, 1925, 1933) first investigated the free torsion problem of a cylinder twisted by forces applied on the lateral surface.

The given cylinder has an arbitrary cross-section R and the boundary curve C , the unit normal to C being denoted by \vec{v} .

Let the system of coordinates be chosen as shown in Figure 2, with the fixed end of the cylinder in the plane $z = 0$, and the free end in the plane $z = l$.

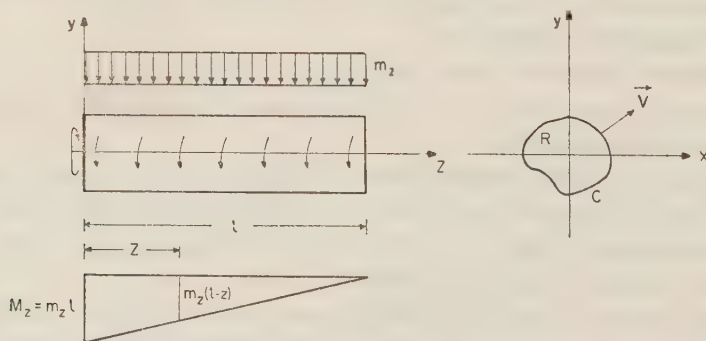


Figure 2

The prescribed surface tractions S_i ($i = x, y$), assumed as independent of z , are directed parallel to the xy -plane, and produce a twisting moment m_z per unit length. Thus, the boundary conditions are:

$$S_z = 0, \quad \int_c S(x, y) ds = 0, \quad \int_c (xS_y - yS_x) ds = m_z \quad (1.5)$$

The relaxed boundary conditions at the ends take the form:

(a) at the end $z = 0$

$$\left. \begin{aligned} \iint_R s_{xz} dA &= \iint_R s_{yx} dA = \iint_R s_{zz} dA = 0 \\ \iint_R (xs_{yz} - ys_{zx}) dA &= m_z \cdot l = M_z \\ \iint_R xs_{zz} dA &= \iint_R ys_{zz} dA = 0 \end{aligned} \right\} \quad (1.6)$$

(b) at the end $z = l$

$$\left. \begin{aligned} \iint_R s_{zz} dA &= \iint_R xs_{zz} dA = \iint_R ys_{zz} dA = 0 \\ s_{zx} &= 0, \quad s_{zy} = 0 \end{aligned} \right\} \quad (1.7)$$

The boundary conditions on the lateral surface are

$$\left. \begin{aligned} s_{xx} \cos(x, \nu) + s_{yx} \cos(y, \nu) &= S_x(x, y) \\ s_{yx} \cos(x, \nu) + s_{yy} \cos(y, \nu) &= S_y(x, y) \\ s_{zx} \cos(x, \nu) + s_{zy} \cos(y, \nu) &= 0 \end{aligned} \right\} \quad (1.8)$$

The problem is to determine a system of tractions s_{ij} which satisfies the equilibrium equations (the body forces F_i are assumed to vanish)

$$s_{ij,j} = 0, \quad (i, j = 1, 2, 3) \quad (1.9)$$

the relaxed boundary conditions (1.6), (1.7), the boundary conditions (1.8), and the Beltrami-Michell compatibility equations.

$$\nabla^2 S_{ij} + \frac{1}{1 + \sigma} \theta,_{ij} = 0 \quad (1.10)$$

where S_{ij} is the stress-tensor, θ its first invariant and σ Poisson's ratio.

Reiner uses the St. Venant semi-inverse method of solution, which consists in making certain assumptions about the stress distribution, leaving sufficient freedom to permit the satisfaction of the differential equations and the boundary conditions.

It is obvious that the torsional couple transmitted across each cross-section is a linear function of the distance from that cross-section to the free end of the cylinder;

hence it is assumed that the stresses in the cylinder have, in part, an appearance similar to those in St. Venant's torsion problem.

As the applied tractions do not vary along the length of the cylinder, it is assumed that the deformation is independent of z , or

$$u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = 0$$

For such deformation, known as plane strain, it is necessary that $s_{zz} = 0$, but as a variable s_{zz} at the end sections may produce a bending or stretching of the cylinder we have to add the bending and torsion type solutions.

The s_{ij} system obtained is

$$\left. \begin{aligned} \frac{s_{xx}}{a'} &= \frac{\varphi}{2} - \frac{\partial^2 \psi}{\partial y^2} \\ \frac{s_{yy}}{a'} &= \frac{\varphi}{2} - \frac{\partial^2 \psi}{\partial x^2} \\ \frac{s_{zz}}{a'} &= -\sigma \nabla^2 \psi - \varphi + A + Bx + Cy \\ \frac{s_{yz}}{a'} &= \frac{1}{2}(l-z)\left(\frac{\partial \varphi}{\partial y} + x\right) \\ \frac{s_{xz}}{a'} &= \frac{1}{2}(l-z)\left(\frac{\partial \varphi}{\partial x} - y\right) \\ \frac{s_{xy}}{a'} &= \frac{1}{4}(x^2 - y^2) + \frac{\partial^2 \psi}{\partial x \partial y} \end{aligned} \right\} \quad (1.11)$$

where

$$\nabla^2 \varphi(x, y) = 0 \quad \text{in } R$$

$$\frac{d\varphi}{ds} = y \cos(x, \nu) - x \cos(y, \nu) \quad \text{on } C \quad (1.12)$$

$$\nabla^4 \psi = 0 \quad \text{in } R$$

$$\left. \begin{aligned} \frac{d}{ds} \left(\frac{\partial \psi}{\partial y} \right) &= \frac{\varphi}{2} \cdot \cos(x, \nu) + \left(\frac{x^2 - y^2}{4} + \frac{S}{a'} \right) \cos(y, \nu) - \\ - \frac{d}{ds} \left(\frac{\partial \psi}{\partial x} \right) &= \frac{x^2 - y^2}{4} - \frac{S}{a'} \cos(x, \nu) + \frac{\varphi}{2} \cos(y, \nu) \end{aligned} \right\} \quad \text{on } C \quad (1.13)$$

$$a' = \frac{\int_C (xS_y - yS_x) ds}{\frac{1}{2} \iint_R (x^2 + y^2 + x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x}) dx dy} = \frac{m_z}{D'} \quad (1.14)$$

Thus, if the torsion function φ is known for R , the value of the derivatives of ψ is known on C , and the problem is reduced to the fundamental boundary-value problem in the biharmonic equation.

2. Explicit formulae are known only for the stress in a beam of elliptical cross-section twisted by constant tangential load S .

The bi-harmonic function $\psi(x, y)$ is

$$\psi(x, y) = \frac{b^2xy^3 - a^2x^3y}{12(a^2 + b^2)} \quad (2.1)$$

where a and b are the semi-axes of the ellipse. The torsion function $\varphi(x, y)$ is

$$\varphi(x, y) = \frac{b^2 - a^2}{a^2 + b^2} xy \quad (2.2)$$

The stress components, following Reiner, take the form

$$\left. \begin{aligned} s_{xx} &= -\frac{a'a^2}{2(a^2+b^2)} xy \\ s_{yy} &= \frac{a'b^2}{2(a^2+b^2)} xy \\ s_{zz} &= -\frac{a'(b^2-a^2)(\sigma+2)}{2(a^2+b^2)} y \\ s_{yz} &= \frac{a'b^2}{a^2+b^2} x(l-z) \\ s_{xz} &= -\frac{a'a^2}{a^2+b^2} y(l-z) \\ s_{xy} &= \frac{a'}{4(a^2+b^2)} (b^2x^2 - a^2y^2) \end{aligned} \right\} \quad (2.3)$$

where

$$a' = \frac{a^2b^2}{4(a^2+b^2)}, \quad m_z = 2S\pi ab$$

An extension of this theory to the problem of constrained torsion is given here for the case of an elliptical section twisted by constant tangential tractions s applied on the lateral surface. Then, making use of the linear relation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.4)$$

we find by integration the following expressions for the displacement components

$$\left. \begin{aligned} u_x &= \frac{a'(1+\sigma)}{2E(a^2+b^2)} \left\{ \frac{1}{2} [-a^2(1+\sigma) + \sigma b^2] x^2y + (a^2+b^2)y(l-z)^2 - \right. \\ &\quad \left. - \frac{1}{3} [a^2 + \frac{1}{2} [b^2(1+\sigma) - \sigma a^2]] y^3 \right\} \\ u_y &= \frac{a'(1+\sigma)}{2E(a^2+b^2)} \left\{ \frac{1}{2} [b^2(1+\sigma) - \sigma a^2] xy^2 - (a^2+b^2)x(l-z)^2 + \right. \\ &\quad \left. + \frac{1}{3} [b^2 - \frac{1}{2} [-a^2(1+\sigma) + \sigma b^2]] x^3 \right\} \\ u_z &= \frac{a'(1+\sigma)}{E} \frac{b^2 - a^2}{a^2 + b^2} xy(l-z) \end{aligned} \right\} \quad (2.5)$$

where the constants of integration representing a rigid body translation and a rigid body rotation can be assumed to vanish by fixing the origin, an element of the z -axis and an element of the xz -plane at the origin, These give six independent conditions

$$u_x = u_y = u_z = \frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial z} = 0 \quad \text{at } (0, 0, 0) \quad (2.6)$$

From Equ. (2.5,3) it can be seen that the longitudinal displacement u_z varies linearly along the cylinder and becomes a maximum at the fixed end ($x = 0$) in the same manner as the torque (Compare Figure 3).

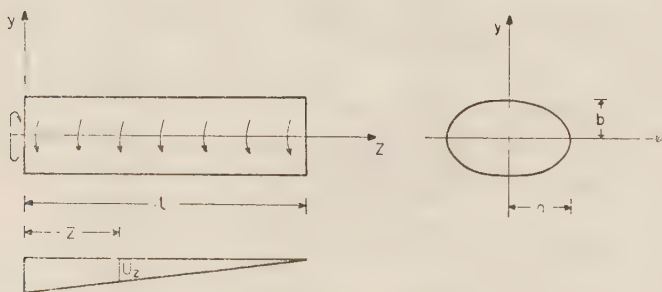


Figure 3

U_z diagram for $xy = \text{const.}$

3. The problem of constrained torsion by forces applied on the lateral surface can now be solved with the help of the three following problems:

- (a) the free torsion, where the bases of the cylinders are subjected to external tractions, following St. Venant;
- (b) the free torsion of the cylinder by forces on the lateral surface, following Reiner;
- (c) the constrained torsion with the lateral surface free from load.

While the solution of (a) is known for many sections, the solution of (b) is only known for an elliptical beam twisted by constant tangential tractions. The problem (c) was only approximated; by Foepl for the elliptical beam, by S. Timoshenko⁷ (1922) for a narrow rectangle and by J. Nowinski⁸ (1954) for a rectangular bar.

Let us construct the solution for the constrained torsion of an elliptical bar twisted by a constant tangential load applied on the lateral surface.

We impose on (b) a moment M_z according to (a), in order to satisfy the boundary conditions $u_z = 0$ at $z = 0$. Its magnitude is

$$M_z = D \cdot \alpha = D\alpha' \frac{1+\sigma}{E} l = D \cdot \frac{4S(a^2+b^2)}{a^2b^2} \cdot \frac{1+\sigma}{E} \cdot l = 2\pi ab l S \quad (3.1)$$

where D , the torsional rigidity of the elliptical cross-section, has the value

$$D = \frac{\pi\mu a^3 b^3}{a^2 + b^2} \quad (3.2)$$

and μ is the modulus of rigidity.

Using Foepl's approximate solution we comply with the relaxed boundary condition at $z = l$ (1.7), not interfering with the already satisfied boundary condition at $z = 0$ ($u_z = 0$), and so the problem (c) is solved.

The s_{ij} system takes the form

$$s_{xx} = -\frac{\alpha' a^2}{2(a^2 + b^2)} xy \quad (3.3)$$

$$s_{yy} = -\frac{\alpha' b^2}{2(a^2 + b^2)} xy \quad (3.4)$$

$$s_{zz} = -\frac{\alpha'(2+\sigma)(b^2 - a^2)}{2(a^2 + b^2)} xy + \alpha'y(1+\sigma)l \frac{b^2 - a^2}{a^2 + b^2} xye^{-\beta z} \quad (3.5)$$

$$s_{yz} = \frac{\alpha' b^2}{a^2 + b^2} x(l-z) - \frac{1}{\beta} e^{-\beta z} \cdot 4Kb^2x(a^2b^2 - b^2x^2 - a^2y^2) \quad (3.6)$$

$$s_{xz} = -\frac{\alpha' a^2}{a^2 + b^2} y(l-z) - \frac{1}{\beta} e^{-\beta z} \cdot 4Ka^2y(a^2b^2 - b^2x^2 - a^2y^2) \quad (3.7)$$

$$s_{xy} = \frac{\alpha'}{(4a^2 + b^2)} (b^2x^2 - a^2y^2) + K(a^2b^2 - b^2x^2 - a^2y^2)^2 e^{-\beta z} \quad (3.8)$$

where

$$K = \alpha' \beta^3 (1+\sigma) l \frac{b^2 - a^2}{16(a^2 + b^2)} \cdot \frac{1}{a^2 b^2} \quad (3.9)$$

$$ab\beta^2 = -\frac{a^2 + b^2}{ab} + \sqrt{\frac{(a^2 + b^2)^2}{a^2 b^2} + \frac{16}{1+\sigma}} \quad (3.10)$$

$$\alpha' = \frac{4S(a^2 + b^2)}{a^2 b^2} \quad (3.11)$$

4. Eqs. (3.3) to (3.11) give expressions for the stress-components in a cylinder of elliptical cross-section, fixed at the end $z = 0$ (so that the cross-section remains plane) and free from all tractions and hindrance at the end $z = l$, loaded by tangential tractions S applied on the sides of the cylinder in planes parallel to the xy -plane and constant along the circumference of the ellipse and along z . The resultant of this loading is equivalent to a total torque $M_z = 2\pi abls$, which is taken up by tractions acting tangentially in the cross-section $z = 0$, and distributed in accordance with (3.6) and (3.7).

When we introduce $a = b = R$ we find Reiner's solution for a circular bar where warping does not occur and no constraint is therefore required to keep the section plane.

Five problems may be indicated for further study:

- (a) explicit formulas for Reiner's problem in the case of sections other than an ellipse;
- (b) exact solutions for the constrained torsion problem approximated by Foepppl, Timoshenko and Nowinski;
- (c) constrained torsion of cylindrical bars by constant forces on the lateral surface in the case of an arbitrary cross-section;
- (d) extending (a) to the case where the applied surface tractions vary along the length of the cylinder;
- (e) extending (c) in accordance with (d).

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SAFETY FACTOR IN STRUCTURES IN THE ELASTIC RANGE WITHOUT STRESS-LOAD PROPORTIONALITY

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ABSTRACT

The paper deals with elastic design of structures having no stress-load proportionality. It is suggested that these structures be designed on the basis of the load producing a critical stress, rather than on the conventional basis of the admissible stresses. The suggested method ensures the intended required safety factor against the appearance of the critical stress (for instance, yield stress).

GENERAL

This paper deals with the design of a group of structures by the conventional elastic method, i.e. on the basis of admissible (working) stresses. Ultimate load design is mentioned only for the purpose of comparison.

In these structures there exists no proportionality between load and stress, in spite of the stress being within the elastic range and the validity of Hooke's law. Examples of such structures are: structures where the change in their geometry due to deformation cannot be neglected in the application of the equilibrium conditions; composite structures; partially precast structures; strengthened structures (beams, columns, foundations); structures erected by stages; partially and totally prestressed structures; structures with thermal and assembly stresses; structures with "conditional restraint", etc.

It is maintained that the design of such structures should be based on the failure load producing a critical stress, rather than on the admissible stress; only this method ensures the intended required safety factor against the appearance of the critical stress (for instance, yield stress) which is the true strength criterion of the elastic design. It should also be noted that in the above structures only a modified superposition method can be applied.

According to the conventional elastic method, the admissible load P_{adm} is the one producing the admissible stress σ_{adm} in the critical section of the structure. Through long routine use, the concept of admissible stress has come to be regarded as an independent criterion in its own right; in fact, however, the admissible stress depends upon two other quantities and equals the critical stress σ_{cr} divided by the safety factor k :

$$\sigma_{adm} = \sigma_{cr}/k \quad (1)$$

We shall not deal here with the numerous problems related to these two quantities. There is no agreed definition of the critical stress. In steel structures the term usually refers to the yield stress σ_y ; in concrete, it is mostly the prism (or cylinder) strength*. The safety factor is governed by a number of factors, but will be used here as defined in existing elastic Codes of Practice — or by definition (1), σ_{cr} and σ_{adm} being given by the Codes.

For simplicity, assume that the structural steel has a yield stress (= proportional limit) of $\sigma_y = 2400 \text{ kg/cm}^2$ and $\sigma_{adm} = 1200 \text{ kg/cm}^2$, $k = 2$ being the safety factor. By the conventional method, P_{adm} would then be the load producing $\sigma_{adm} = \pm 1200 \text{ kg/cm}^2$ in the steel structure. On the other hand, the failure load P_y is the one producing the yield stress $\sigma_y = \pm 2400 \text{ kg/cm}^2$ at the same point; hence the admissible load P_{adm}^* (the asterisk added for distinction from the conventional P_{adm}) should be:

$$P_{adm}^* = P_y/k \quad (2)$$

In the ordinary cases, with σ proportional to P , $P_{adm}^* = P_{adm}$; but when no such proportionality exists, $P_{adm}^* \leq P_{adm}$. In these cases it is suggested that the design be based on P_{adm}^* . This method is illustrated by a number of examples.

This paper does not deal with cases where the equivalent stress is determined in accordance with the different strength theories. It is also assumed that problems of stress concentration and fatigue do not arise, and that limited dynamic loading is expressed by an equivalent statical loading.

EXAMPLES

Figures 1 and 2 show two well-known cases, where equilibrium conditions should be applied to the *deformed structure*. Figure 1a shows two horizontal bars connected by hinges and supporting a vertical load P . The load-deflection curve is concave, and so is the load-stress curve, given in Figure 1b (for $A = 1 \text{ cm}^2 = \text{const}$; $E = 2.1 \times 10^6 \text{ kg/cm}^2$):

$$\sigma = \frac{1}{2} \sqrt[3]{\frac{P^2 E}{A^2}} \quad (3)$$

It can be seen that:

$$P_{adm}^* = 115 \text{ kg} > P_{adm} = 81 \text{ kg}$$

The existing (working) stress produced by P_{adm}^* is $\sigma^* = 1510 \text{ kg/cm}^2 > \sigma_{adm}$, but a safety factor of 2 against the appearance of σ_y is still ensured; for such a reliable material as steel, even higher working stresses should raise no objection.

Figure 2a shows a slender bar subjected to two equal and opposite eccentric forces P . Here the secant formula applies (Figure 2b):

* In buckling, it is the critical buckling stress.

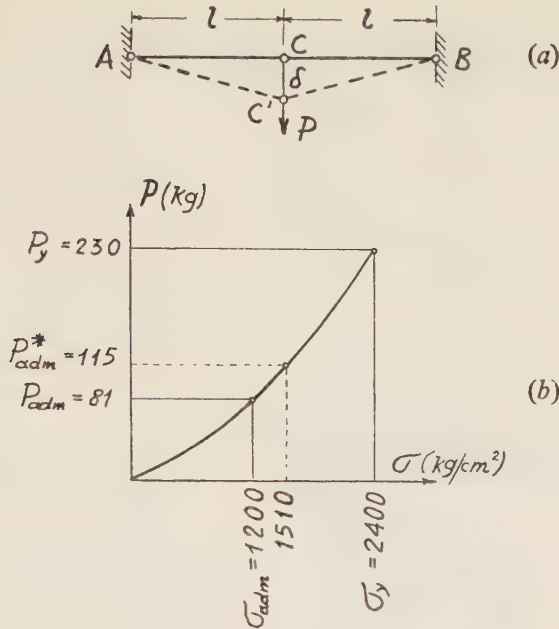


Figure 1

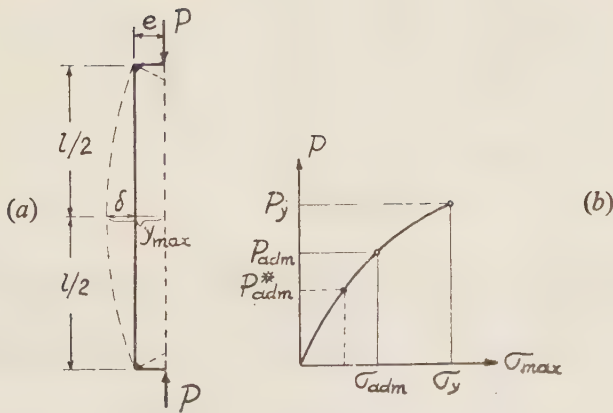


Figure 2

$$\sigma_{max} = \frac{P}{A} + \frac{Pe \sec\left[\frac{1}{2}l\sqrt{P/(EI)}\right]}{Z} \quad (4)$$

The curve is convex and design by σ_{adm} (or P_{adm}) would not provide an adequate safety factor against σ_y .

A simple model of a prestressed structure (Figure 3a) consists of a short pipe ($A_{II} = 1 \text{ cm}^2$) with a bolt ($A_I = A_{II} = 1 \text{ cm}^2$) passing through it. By turning the nut,

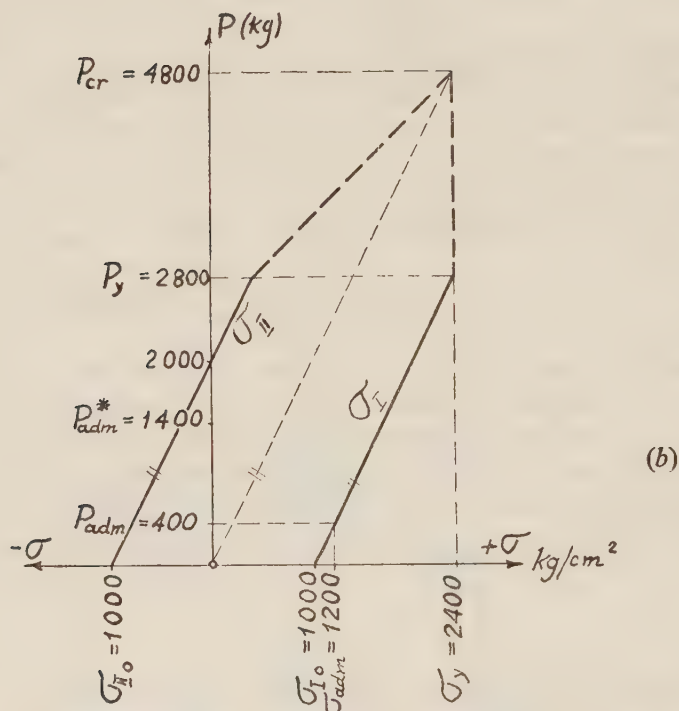


Figure 3

the bolt is pretensioned ($\sigma_{I_0} = 1000 \text{ kg/cm}^2$) and the pipe precompressed ($\sigma_{II_0} = -\sigma_{I_0} = -1000 \text{ kg/cm}^2$). We now determine the admissible axial tensile force P which may be applied to the pipe. Figure 3b shows the P/σ_I and P/σ_{II} lines*:

$$P_{adm}^* = 1400 \text{ kg} \geq P_{adm} = 400 \text{ kg}$$

We could also have made our point, of course, by prestressing up to $\sigma_{I_0} = -\sigma_{II_0} = 1200 \text{ kg/cm}^2 = \sigma_{adm}$; in that case: $P_{adm} = 0$ but $P_{adm}^* = 1200 \text{ kg}$.

The dashed lines in Figure 3b show the extension into the plastic range. The ultimate load P_{cr} of the structure does not depend on the prestress:

$$P_{cr} = A_I \sigma_y + A_{II} \sigma_y = 2 \times 1.0 \times 2400 = 4800 \text{ kg}$$

* Up to the point where the pipe's prestress is reduced to zero ($P=200 \text{ kg}$), the force P could have been applied directly to the bolt without affecting the P/σ lines.

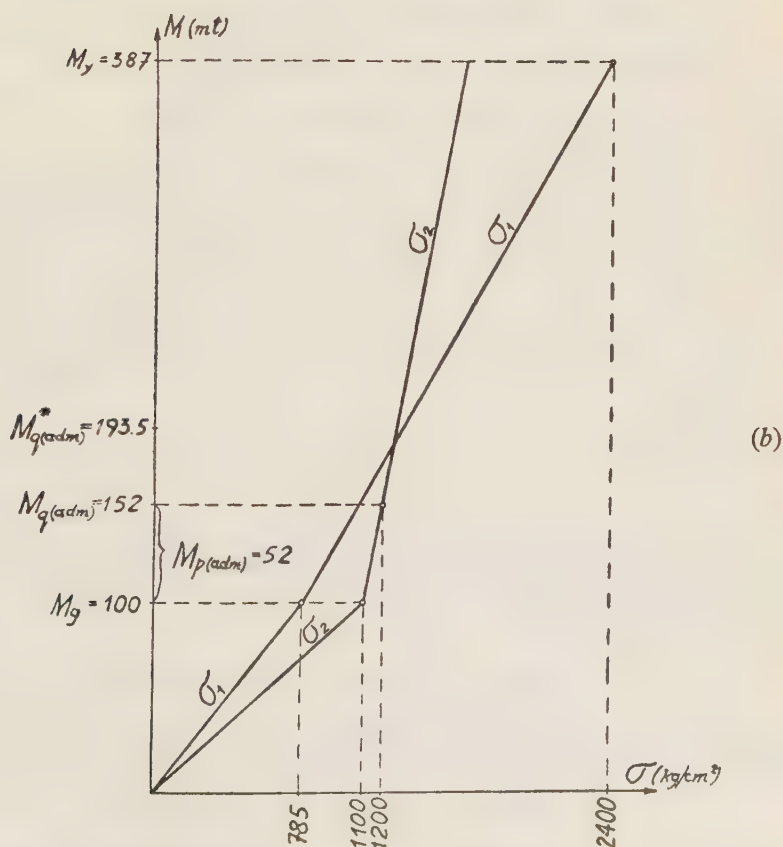
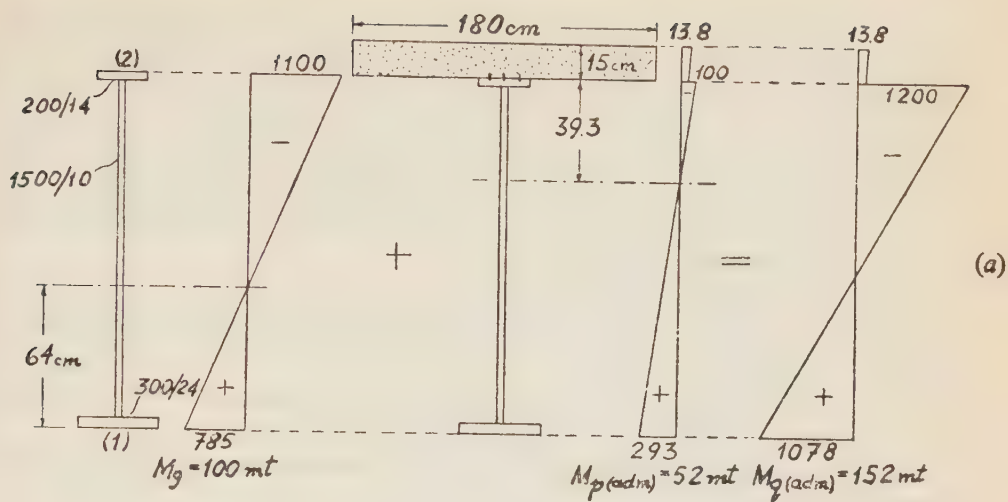


Figure 4

A composite steel-concrete beam is shown in Figure 4a. This is one of the longitudinal beams supporting a bridge with a concrete roadway. The idea is to use the steel beam ($I = 813,000 \text{ cm}^4$) for carrying the dead load in the first stage of erection (bending moment M_g). In the second stage, the composite beam ("transformed section"; $I_{eq} = 2,046,000 \text{ cm}^4$) carries the additional live load including pavement etc. (M_p). Special welded dowels ensure the connection between the concrete slab and the steel section. The lower flange of the steel section is made larger in order to achieve better utilisation of both flanges.

Assuming the bending moment of the first stage to be $M_g = 100 \text{ mt}$, we determine the additional admissible bending moment M_p which may be carried by the composite beam at the second stage.

The stress distributions which lead to $M_{p(adm)}$ and to $M_{q(adm)} = M_g + M_{p(adm)}$ by the conventional method are shown in Figure 4a*.

The M/σ lines for the extreme fibres of the steel section are shown in Figure 4b, which also reveals:

$$M_{q(adm)}^* = M_y/k = 387/2 = 193.5 \text{ mt} > M_{adm} = 152 \text{ mt}$$

$$M_{p(adm)}^* = M_{q(adm)}^* - M_g = 93.5 \text{ mt} \gg M_{p(adm)} = 52 \text{ mt}$$

The (working) stresses produced by $M_{q(adm)}^* = 193.5 \text{ mt}$ are $\sigma_1 \cong 1310 \text{ kg/cm}^2$ and $\sigma_2 = 1280 \text{ kg/cm}^2$.

When actual loads are expected to exceed design loads, *strengthening* of existing structures may become necessary. Examples are: steel joist with additional welded flange-plates; truss-strengthened beam; arch with tie-rod, etc.

Figure 5 shows a symmetrical model with "conditional restraint" consisting of three steel bars ($A = 1 \text{ cm}^2 = \text{const.}$) supporting a vertical load P . The bar X comes into action only after contact has been established at B ($P = P_t$). The additional load $P - P_t$ is then resisted by all three bars, X resisting twice as much as Y (for the given slope of 45°).

Figure 5 also shows the relationships P/σ_X and P/σ_Y . The set-off continuous lines represent the case $t = 0$, the set-off dashed lines showing the extension into the plastic range. The change of t permits the regulation of the stresses in the bars. In this example it is possible to utilise simultaneously the admissible stress in both bars when t corresponds to an initial stress of $\sigma_{Y_t} = 600 \text{ kg/cm}^2$ (continuous lines); in this case we arrive at:

$$P_{adm}^* = 4940/2 = 2470 \text{ kg} < P_{adm} = 2895 \text{ kg}$$

If, on the other hand, t is made to produce $\sigma_{Y_t} = 800 \text{ kg/cm}^2$ (point b ; dot-dash lines), P_{adm}^* is governed by bar X , while P_{adm} is determined by bar Y :

$$P_{adm} = 5230/2 = 2615 \text{ kg} > P_{adm} = 2495 \text{ kg}$$

* Since the stresses in the concrete slab are quite low, we are justified in neglecting the creep of the concrete (for M_g) and also the non-proportional stress-strain relationship.

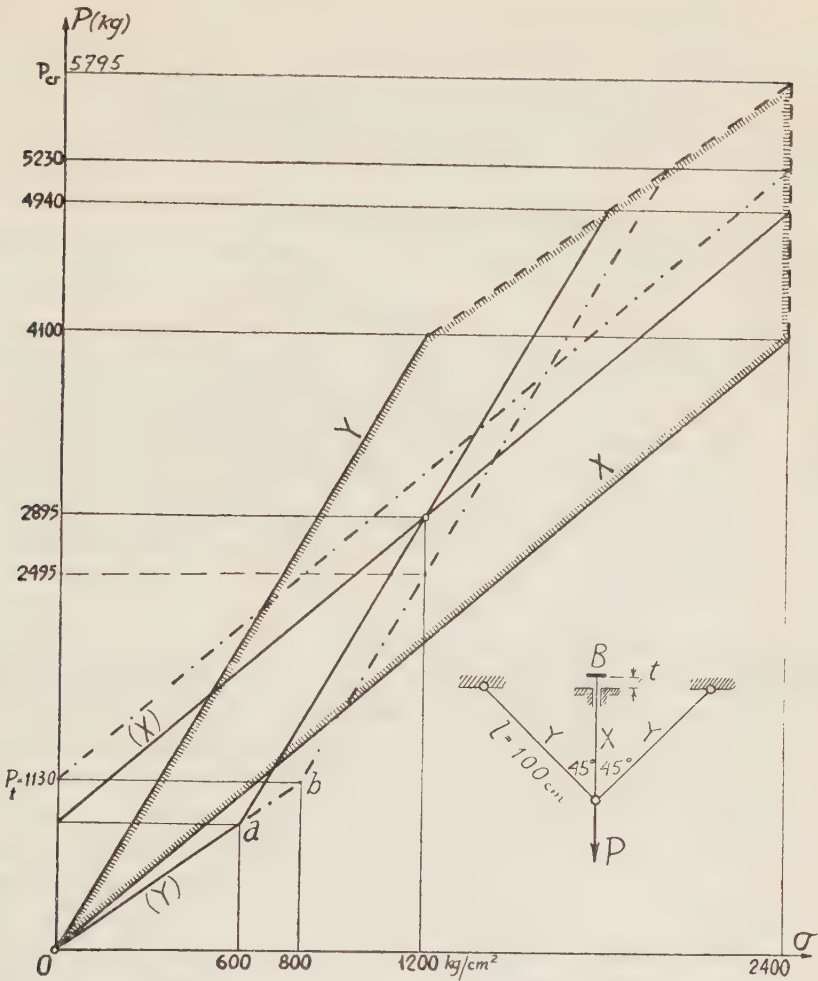


Figure 5

A structure with conditional restraint can also be considered as *partially prestressed* (bar Y having initial stress before X comes into action). Another example: a pipe (subjected to internal radial pressure) inside a non-tightly fitting sleeve. Partial prestressing is especially used in bridges. Consider, for instance, a three-span beam (Figure 6), the middle span BC being usually much longer than the other two. In order to reduce the moment at m it is advantageous to let the end spans AB and CD act as freely overhanging cantilevers in the first stage and establish contact with the outer supports A and D in the second stage only.

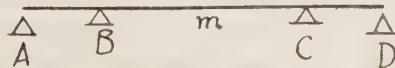


Figure 6

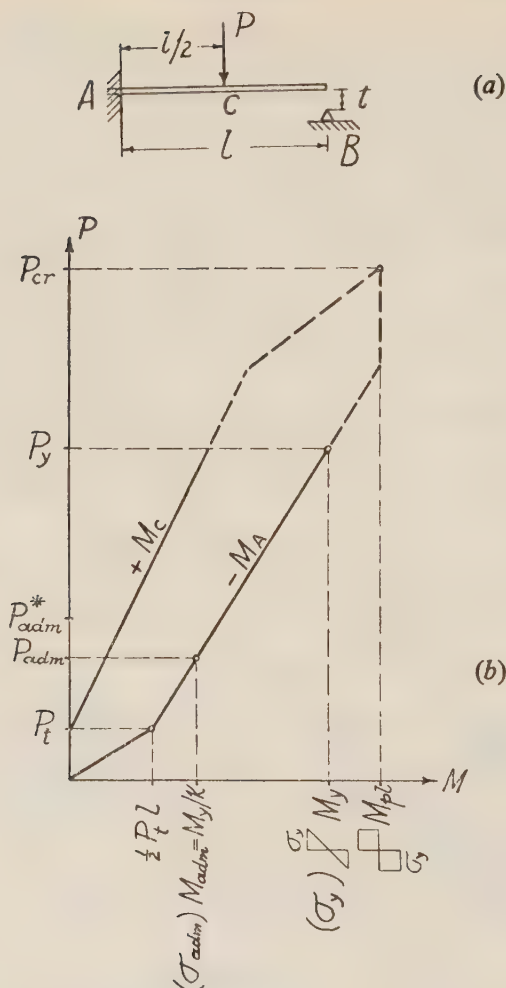


Figure 7

The simpler case of a cantilever with a conditional support B is illustrated in Figure 7. The P/M lines also represent the P/σ_{max} lines up to $M = M_y$. The dashed lines show the extension into the plastic range, neglecting the curvature of the elasto-plastic stage.

CONCLUSION

Non-proportionality between stress and load in the elastic range has been illustrated in a number of examples.

The relation may be represented by graphs of different shapes — curve (concave or convex), straight line not passing through origin, broken straight line, etc. In the majority of cases design by $P_{adm}^* = P_y/k$ entails considerable economy, but there are also opposite cases. Therefore each case should be studied in order to ensure the required safety factor.

EXPERIMENTS ON AERATION METHODS DURING MYCOLOGICAL CITRIC ACID FERMENTATION

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ABSTRACT

In order to create better contact between air and the fermenting media during the production of citric acid by submerged fermentation, three different methods of aeration were tried:

- a) Fermentation on artificially-produced foam.
- b) Fermenting while spraying the medium into air.
- c) Aeration of the fermenting medium with the aid of a special vibro-mixer.

Special fermentors were assembled in each case and series of runs were tried with sucrose solutions and diluted molasses.

Of the three methods only the last one, using the vibro-mixer, gave positive results, reducing the time of fermentation and achieving quite a high rate of conversion.

INTRODUCTION

The first trials in producing citric acid by the mycological method, although unsuccessful at the time, actually began 65 years ago following the publication of Carl Wehmer's¹ first observations, namely, that certain microorganisms isolated from the air produced this acid from sugar. But only in the late thirties of the present century this method became an important factor in the manufacture of citric acid. To-day it has practically replaced the old Scheele method of producing citric acid from lemon juice, and in the U. S. A. alone the annual output of the biological product has reached a total of over 25,000 tons.

Important advances have accompanied the development of this industry. These have been ably summarised in a number of articles^{2,3}. The original species of microorganisms described by Wehmer were replaced by more productive fungi, such as *Aspergillus niger*⁴ and *Asp. wentii*⁵. Fermentation was originally, and still is in many cases, carried out in shallow pans⁶ and only recently various experiments in submerged fermentation were attempted^{7,8}. As the development of these fungi requires an abundant supply of air, and as ordinary aeration procedures were inadequate, the shallow-pan method was usually preferred in spite of the fact that it entails a large initial outlay and excessive maintenance charges. This method of producing citric acid is also comparatively slow (8 to 10 days) and is, therefore, open to danger of contamination by undesirable microorganisms.

Many attempts have been made to provide the necessary aeration by such means as increasing the surface exposed to the air by soaking wood shavings, or similar porous material, in the medium to be fermented⁹.

In the present work three attempts have been made to overcome the difficulties of aeration, and although they were not entirely successful it was considered worthwhile to describe them herewith. Studies of submerged fermentation of citric acid published to date are all based on one of two methods, namely the shaking flask method or the column-fermentor, or on modifications of both. The object of the present work, however, was to study three other possibilities:

- a) Fermentation on artificially produced foam.
- b) Fermenting while spraying the medium into the air.
- c) Aeration of the fermenting medium with the aid of a special vibro-mixer.

The working theory underlying all three of these methods was to permit better contact between air and the fermenting media, thus reducing the period of active fermentation and preventing the development of undesirable microorganisms.

EXPERIMENTAL

A. Fermentation on foam

Cahn⁹ reported considerably accelerated rates of fermentation by *Aspergillus niger* grown on solid materials, such as wood shavings or beet pulp, impregnated with the medium. Such accelerated rates were no doubt due to greater contact surfaces between the three phases: fungi, medium and air. The final products of fermentation were then extracted by diffusion. On the other hand, it is well known that liquids used as media in this process are apt to produce considerable quantities of foam during aeration. In industry such foam is combated by various means, such as addition of anti-foam or mechanical foam-breakers.

The idea was to utilise the foam as a bed for this fermentation, and grow the fungi on the enormous surface created by it, thus dispensing with the addition of a solid phase and consequently also eliminating the necessity of extracting the citric acid.

The apparatus used (Figure 1) consisted of a large fermenting vessel A, only partially filled with fermenting medium. The air entering through B facilitated the formation of copious foam which was normally trapped in vessel D, broken up mechanically and returned to the main vessel while the air escaped through F.

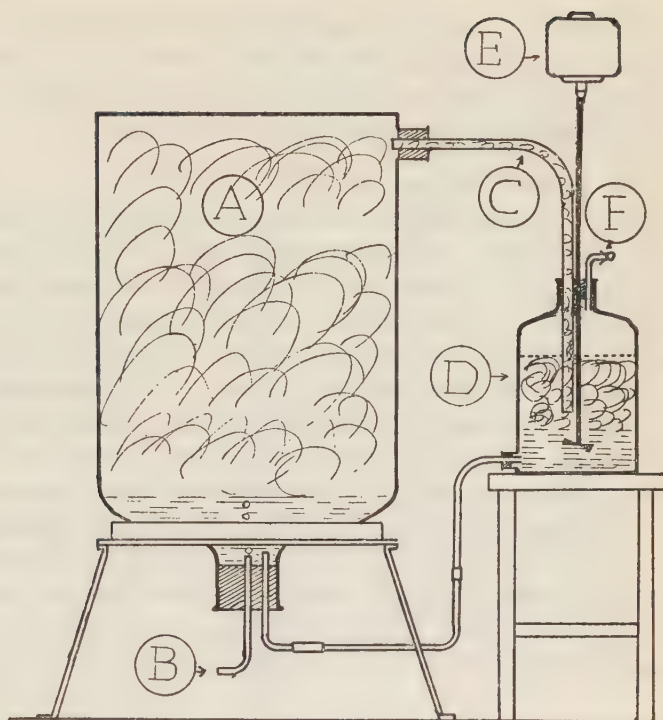
In all experiments by this method the medium consisted of: 160 g sucrose, 1.0 g KH_2PO_4 , 2.5g NH_4NO_3 , 0.25g $\text{MgSO}_4 \cdot 7\text{H}_2\text{O}$, the whole made up to 1 litre with distilled water. The medium was sterilised at 15 psig for 15 minutes and the acidity adjusted to pH 4.0 with HCl. As foaming agent, white of egg was used. Before each run the whole apparatus was sterilised with a 10% solution of formalin, washed with hot water and detergent, rinsed several times with hot water and finally twice with sterile distilled water.

Inoculum used: 20ml of a heavy suspension of spores of the strain *Aspergillus niger* ATCC 1015 cultivated on a solid medium of sugar-salt-agar similar to Czapek's medium, four transfers having been made before final sporulation.

Figure 1.

Foam fermentor assembly

- A - foam chamber
 B - air inlet
 C - foam run-over
 D - foam breaker
 E - motor
 F - air outlet.



After inoculation, aeration was started at a rate of 100 ml per min. Almost immediately a permanent foam built up and continued to circulate slowly. Samples were drawn every 24 hours and tested microscopically as well as for total acidity and residual sugar.

Four identical runs, six days each, gave negative results as shown in Table I, which represents one typical run. Practically no acid was formed and even the growth of fungi was very poor: formless flakes were found in the liquid below the foam.

TABLE I

Time hrs.	Acidity % citric acid	Appearance under microscope	Residual sugar % sucrose
0	—	clear	14.0
24	—	"	14.0
48	—	"	14.0
72	—	Germination sets in	14.0
96	0.3	Non-uniform mycelial flakes	13.0
120	0.3	" "	13.0
144	0.6	" "	12.0

The failure of the above experiments was explained by the possibility that the presence of the foam impedes the development of the mycelium; it was assumed that with the use of previously prepared mycelium, added to the medium in finely disintegrated form, better results might be obtained.

The mycelium was then prepared as follows:

100 ml of the sugar-salt medium were inoculated with a suspension of spores prepared as described above. After 5 days' incubation at 28°C, a continuous compact mycelium was formed without sporulation. The liquid was decanted and the mycelium, washed with 3 portions of sterile distilled water, was transferred to a Waring Blender together with the substrate intended for the fermentation. Disintegration produced a fine suspension of the mycelium in the medium. This suspension was used as inoculum to 1 litre of substrate, all conditions of this experiment being the same as above. As control, 100 ml of the substrate were inoculated with a small portion of the disintegrated mycelium and left for incubation at rest. However, while in the stationary control flask a new continuous mycelium was formed and after 6 days 4.4 g of citric acid were present, the main experimental batch again proved unsuccessful: the disintegrated mycelium particles agglomerated into formless flakes and failed to rise with the foam. Practically no sugar was used up by the fungi and no acid formed. A duplicate test under similar conditions was equally unsuccessful.

B. Fermenting while spraying the medium into air

This experiment was based on the assumption that better results might be obtained by spraying the medium into air, as is done, for instance, during hydrogenation of oils, in which the liquid phase is sprayed into the gaseous phase.

Figure 2 shows the specially-designed sprayer, as used in the apparatus shown in detail in Figure 3. The liquid in flask A was drawn by sprayer B, through which a large volume of air was passed, distributing the medium in the form of a fine mist of droplets. The mist-carrying air escaped through a series of cyclone-shaped traps (E) in which the droplets were separated and returned to the fermenting vessel. At the same time fresh air was introduced through pipe C in order to agitate the liquid.

The following procedure was adopted:

One litre of the sterilised sugar-salt medium was transferred to vessel A. After inoculation with a suspension of spores, aeration through pipe C began at the rate of 500 ml per minute. Aeration was maintained for 48 hours, when germination set in and small particles of mycelium were formed, giving the entire liquid an appearance of a cloudy suspension. At this point spraying was started, with a discharge rate of at least 21 litres of air per minute. Sterile water was constantly added, to make up losses due to evaporation.

The results obtained in 5 runs are given in Table II.

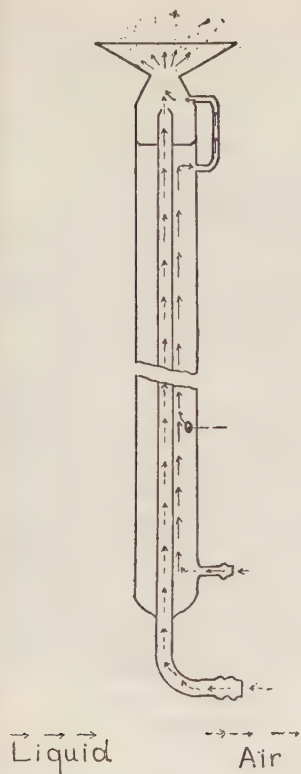


Figure 2.
Sprayer

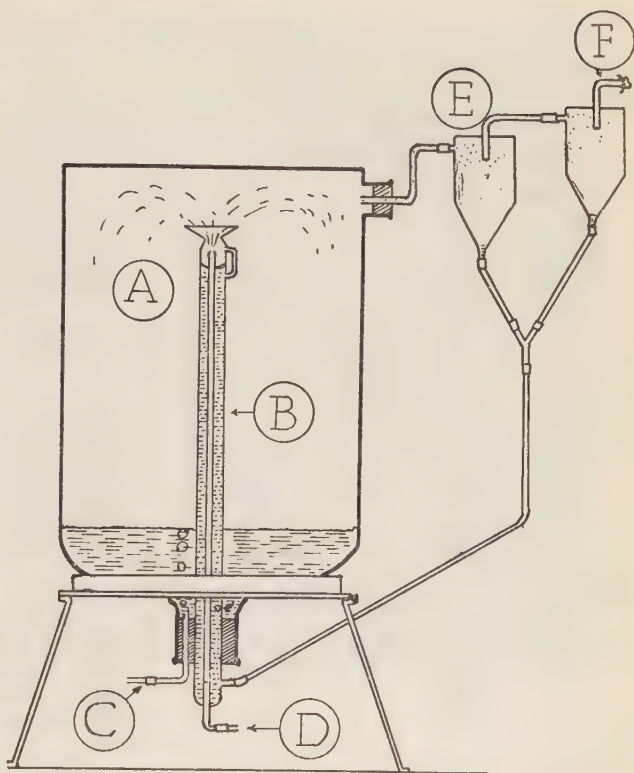


Figure 3.

Spray-fermentor assembly

- | | | |
|----------------------|-------------|-------------------|
| A - spray chamber | B - sprayer | C - agitating air |
| D - air for spraying | E - traps | F - air outlet |

C. Aeration through vibro-mixing

The use of a vibro-mixer, practised in chemical laboratories, was recently adopted also in biotechnological work. Such vibro-mixers are manufactured at present in several industrial sizes. In these experiments, a laboratory 40 W model of Swiss make was used. This mixer contains no rotating parts; instead, a large iron armature is attracted by an electromagnet supported by springs (see Figure 4), giving rise to rapid vibrations in synchronism with the electric current (in this case, 50 cycles per second), the amplitude of the vibrations being too small to be observed visually. When the vibro-mixer is used in conjunction with a special aeration tube, the entering air is finely dispersed in the form of minute bubbles. As far as known, such vibrators have not yet been used in citric acid fermentation. Figure 5 shows the apparatus used in the following runs.

Before starting the experiments on molasses, trial runs were made on model sugar-salt solutions containing, per litre, 170 g sugar, 2.5 g NH_4NO_3 and sufficient

TABLE II

Time hrs.	Acidity % citric acid	Residual sugar % sucrose	Aeration ml/min	Remarks
Run No. 1				
24	—	—	500	Germination sets in
48	—	—	500+21,000	Small units of mycelium form
72	0.22	12.7	500+21,000	
96	0.64	11.2	500+21,000	Mycelium agglomerates into large clumps
120	0.70	11.2	500+21,000	System clogged
Run No. 2				
24	—	—	500	Germination sets in
48	—	—	1000+21,000	Mycelium develops
72	0.30	12.0	1000+21,000	Mycelium develops
96	0.98	11.7	1000+21,000	Mycelium develops
120	1.70	10.9	1000+21,000	Mycelium develops
144	2.80	9.3	1000+21,000	System clogged
Run No. 3				
24	—	—	500	Germination sets in
48	—	—	1000+21,000	Mycelium develops
72	0.18	13.3	1000+21,000	Mycelium develops
96	0.20	13.0	1000+21,000	Mycelium develops
120	0.18	12.4	1000+21,000	System clogged
Run No. 4				
24	—	—	500	} No mycelium formed, liquid remains clear
48	—	—	500	
144	—	—	500	
Run No. 5				
24	—	—	1000	Germination sets in
48	—	—	1000+21,000	Mycelium develops
72	0.32	12.0	1000+21,000	
96	0.83	11.6	1000+21,000	
120	1.05	11.0	1000+21,000	
144	2.8	9.3	1000+21,000	
168	2.8	7.0	1000+21,000	System clogged, strong odour of alcohol

HCl to bring the solution to pH 2.0. Aeration and mixing were started immediately upon transfer of the hot solution into the fermentor. After cooling, the solution was inoculated as usual and the aeration rate brought up to 500 ml per min. Acidity was measured once a day. Fermentation ended after 5 days, with a yield equivalent to 48% conversion.

In the last stage of these experiments, sugar-beet molasses were used as a source of carbohydrates. Their composition was as follows (Table III):

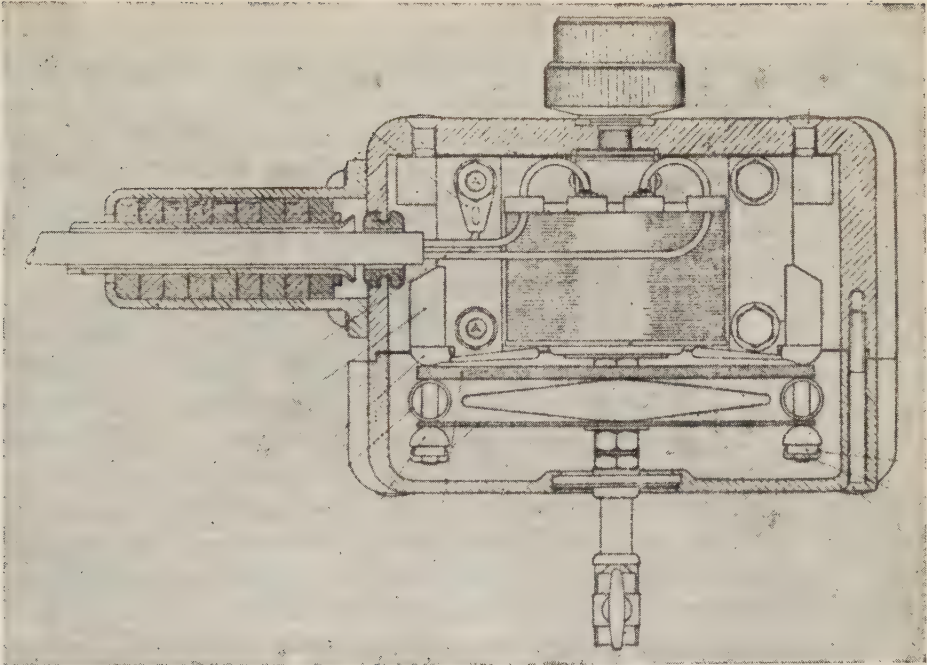


Figure 4.

Vibro-mixer

(A. G. für Chemie-Apparatebau, Zürich)

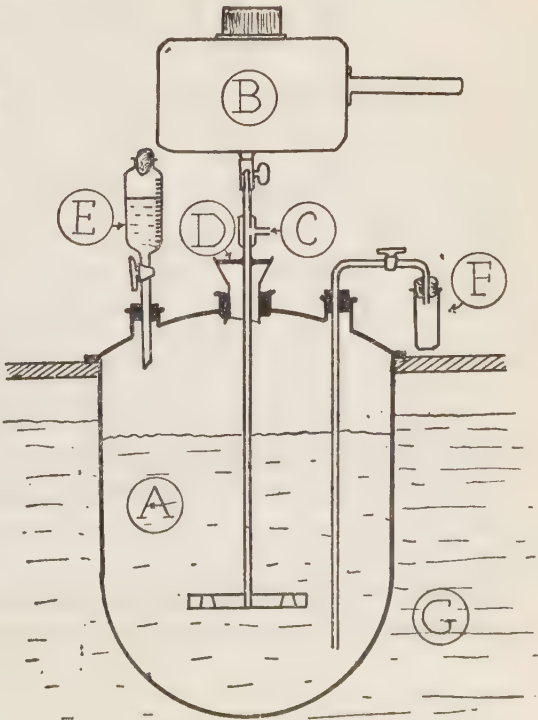
TABLE III

Specific gravity	1.455
Water	17.5%
Total solubles	78.7%
Non-solubles	3.8%
Total sugars	53.0%
Ash	6.1%
Fe ₂ O ₃	0.0488%
P ₂ O ₅	0.0894%

Figure 5.

Fermentor assembly with vibro-mixer

A - fermenting vessel, B - vibro-mixer
C - air inlet, D - diaphragm seal
E - anti-foam receptacle F - sampler
G - constant-temperature bath
(all necks cotton-plugged).



Treatment of the molasses before fermentation consisted in their dilution to 22° Brix (approximately 15% sugars), sterilisation at 120°C for 20 min. and precipitation with potassium ferrocyanide (0.6 g/l).

The diluted molasses were used for propagation as well as for the citric acid fermentation. For propagation purposes, 100 ml batches of diluted molasses were seeded with 1 ml of a suspension of 4–5 days old spores after fourth transfer to sugar-salt-agar slants. Growth was achieved in flasks, by the shaking method, at 28°C. After 2–3 days very small pellets (0.05–0.1 mm) were obtained.

The entire mycelium, together with the solution in which it had been obtained, was used as inoculum for 1 litre batches of diluted molasses in the fermentor. Before inoculation, the solution was aerated at increasing rates, from 10 to 500 ml per min. per litre. This stage was accompanied by considerable foaming, and the rate of aeration was increased further after the foam had subsided each time. Although this preliminary aeration helped to check the formation of foam during the subsequent stage, it was still found necessary to add, after inoculation, about 10 ml of a 2% solution of stearyl alcohol in paraffin oil to prevent excessive foaming during the first ten hours of fermentation. During the fermentation stage, the rate of aeration was 500 to 600 ml per min., at a constant temperature of 28°C.

Samples for analysis were drawn every 24 hours, with the following results.

TABLE IV

Run No.	Duration of fermentation days	Initial sugar concentration %	Final sugar concentration %	Acidity % citric acid	Weight of mycelium g/100 ml	Conversion %	Yield, based on total sugar %
1	5	15	3.3	9.80	2.0	83	65.5
2	6	15	3.3	10.00	1.3	91	66.5
3	5	15	3.8	10.5	1.5	93.5	70.0

DISCUSSION OF RESULTS

1. The foam method

Contrary to expectations, *Aspergillus niger* did not develop on foam. This can probably be explained now as due to one or both of the following causes:

a) This fungus is not a monocellular microorganism, and in order to be capable of metabolic activity the mycelium should be composed of a certain minimum number of cells. In other words, there should probably be a certain minimum thickness of the mycelium unit (partly immersed in liquid) which the foam film is too thin to support.

b) Surface tension forces may also play an important role here, and are highly likely to preclude any metabolic exchange between the cell and the foam film.

2. The spray method

This was actually abandoned for technical reasons without conclusive results. In the first place, it was found impossible to prevent the formation of agglomerations of mycelium (due to insufficient agitation of the liquid by air alone) which clogged the system after four to six days.

Secondly, the spraying process itself required such large amounts of air, that some of the runs failed owing to excessive pressure in the system.

The results of runs 1, 2 and 5 show, however, that the method is feasible in principle, although the system is somewhat cumbersome and requires a fermentor with a capacity of 15 to 20 times the volume of the substrate. Since fermentation by this method could not be accelerated as expected, it was finally abandoned in favour of the vibro-mixer.

3. The vibro-mixer method

The vibro-mixer proved extremely effective and gave quite satisfactory results. Both when using sugar solutions as well as molasses, the results compared favourably with those reported in literature, and experiments on a larger scale should, in the opinion of the authors, prove rewarding. The time of fermentation was reduced to 5 days as against 10–11 days by the shallow pan method, and the rate of conversion was high. The only difficulty encountered was in the formation of foam immediately after inoculation, but this can no doubt be avoided.

CONCLUSIONS

To sum up, it seems justifiable to conclude that:

- 1) Growing of *Aspergillus niger* on the surface of a foam film seems to be impracticable.
- 2) The method of spraying the inoculated medium into air may be feasible as a means of better aeration, but does not accelerate fermentation.
- 3) The method of aeration and subsequent stirring with the aid of a vibro-mixer seems to be both convenient and practicable, reducing the duration of the process and achieving quite a high rate of conversion. Larger-scale experiments will be necessary, however, before this method can be properly evaluated.

ACKNOWLEDGMENT

The authors wish to express their indebtedness to Professor M. Aschner of this Division for his kind assistance during this work and his helpful suggestions and criticism.

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LETTERS TO THE EDITOR

A second order effect in plasticity.

O. ISHAI AND M. REINER, *Rheological Laboratory, Technion—Israel Institute of Technology, Haifa*

Williamson¹ found that when a cylinder of clay is rolled between two plates, a tube is formed. Egan and Jobling² repeated the experiment using different materials and found that the phenomenon is present with plasticine but is not present in polyisobutylene and silicone bouncing putty. We have used (1) cement with a water cement ratio of 26 percent (Figure 1), (2) cement mortar with a water cement ratio of 28 percent and a volume concentration of 42 percent of (standard Leighton Buz-zard) sand (Figure 2). In both cases tubes could be formed. In the second case pronounced radial cracks appeared.

We suggest that this is a second order effect in the plastic deformation of *solids*. Polyisobutylene and silicone bouncing putty are liquids, a solid being a material which has a yield point below which it does not flow. It seems that only granular materials are plastic solids. This would include polycrystalline metals where the single crystals form the granules.

A second order theory will have to account for radial outward flow which manifests itself in both, the formation of a void at the centre and the appearance of circumferential tension causing radial cracks.

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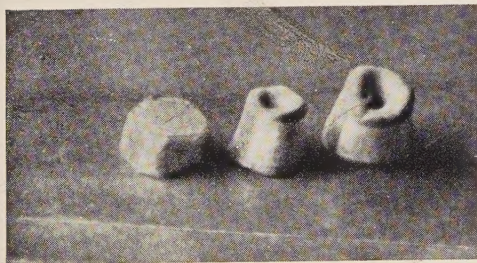


Figure 1

Solid cylinder of cement paste shown on the left, rolled into tubes as shown on the right.

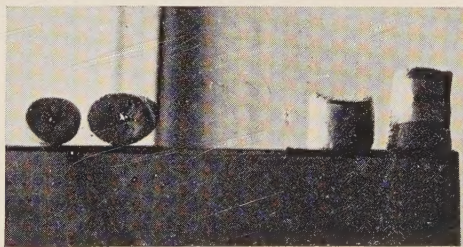


Figure 2

Two tubes rolled from mortar paste. On the left view along axis, on the right—from side.

Received January 9, 1958.

Analysis of thin rectangular plates, which are loaded by transverse concentrated forces

Z. HASHIN, *Division of Mechanics, Technion—Israel Institute of Technology, Haifa.*

In the analysis of thin rectangular plates of elastic material, which are loaded by point forces, the loading is in general considered as a limiting case of either a surface load, distributed over a small rectangular or circular area, or a line load on a small interval.

The solution is found in the form of infinite series. While the series, giving the deflection of the plate, converge quite rapidly, those by which the bending and twisting moments and shearing forces are expressed, converge very slowly. The convergence becomes ever worse as we approach the point of application of the load. Numerical evaluation of moments and shearing forces expressed by such series is all but impossible.

This difficulty may be observed by considering the solution as being composed of two parts. The first part is the solution of an infinite strip, loaded by a point force, which may be expressed in closed form by a method due to Nadai¹; the second part is an infinite series which depends on the boundary conditions which have to be fulfilled in the specific problem under consideration.

The bending moments and shearing forces may thus be given by a closed expression plus a very rapidly converging series, the convergence of which *improves* when approaching the point of application of the load. Numerical results may thus be obtained very easily for all points of a plate.

The mathematical details will be published elsewhere.

REFERENCE

1. NADAI, A., 1925, *Elastische Platten*, Berlin.

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